THE

MATHEMATICAL GAZETTE

The Journal of the Mathematical Association

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THE MATHEMATICAL ASSOCIATION

AN ASSOCIATION OF TEACHERS AND STUDENTS OF ELEMENTARY MATHEMATICS

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'I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto.' BACON

Editorial and Advertising correspondence relating to the Mathematical Gazette should be addressed to the Editor.

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WEEK-DAYS AND MATHEMATICS

BY J. MAYR

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There are two methods of this kind. For the first a calendar is required, the second goes without a calendar. You will, of course, have to keep in mind some rules, but in many cases these rules are more than simple. For instance, if you want to know the week-day of the date 18–9–1456, you put together these three numbers, you add zero and divide by 7. The remainder of this division is the number of the day of the week, beginning with Saturday which is 0. Sunday is 1, Monday 2, Tuesday 3, Wednesday 4, Thursday 5, Friday 6. To resume the above example, you say $18914560 \div 7$, remainder 0, Saturday. You find the day of the week for the date 20–3–1208 by $20312080 \div 7$, remainder 5, Thursday.

THE CALENDAR METHOD

If you want to know the week-day of a date in a certain year, you look first for the week-day of Christmas of that year. Then you can easily find the week-days of all dates with the aid of any calendar.

How to get the week-day of Christmas? In the Julian Calendar you add zero to the year and divide by 7, if the year is a leap-year; if it is a common year you add the difference between the preceding leap-year and the given common year, to the leap-year multiplied by 10. In the case of 1456 you put 14560, for 1457 you put 14561, for 1458 you put 14562, etc. Algebraically you find the week-day c of Christmas for the year Y, if $Y \div 4$ has the remainder r, by the formula $[10(Y-r)+r] \div 7$, remainder c.

If you want to know the week-day of 1456 July 20th, you see in the calendar of 1958 that Christmas is on a Thursday. In 1456 Christmas was on a Saturday, two days later on. July 20th is in the calendar of 1958 on a Sunday, so it must be on a Tuesday in 1456, also two days later. This conclusion you draw for dates from March 1st. For January and February you look for the week-day of Christmas of the preceding year, which is the same as that of the following New-Year, to avoid confusion that could eventually be caused by a 29th of February.

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This method can be proved in the following way. You learn by week-day tables, that in the year 1 the New-Year fell on a Saturday. Then the 25th of December of the year 0 fell also on a Saturday. If there were no leap-years you should find the week-day c of Christmas in the year Y by $Y \div 7$, remainder c, for the week-day of every date advances by one day every year, the remainder after the division $365 \div 7$ being 1. In the leap-years it advances by 2 days. You find the number p of leap-years up to Christmas of the year Y by $Y \div 4 = p$, remainder r, and $p = (Y - r)/4$. So you find c of the year Y by the formula

$$\left(Y + \frac{Y-r}{4}\right) \div 7, \text{ remainder } c.$$

Remember that the apparent fraction $(Y - r)/4$ is in reality a whole number, for you have taken away the remainder r from the dividend Y, and therefore you are permitted to add a multiple of 7, for instance $7(Y - r)/4$, to the dividend, without changing the remainder r. So you have

$$\left(Y + \frac{8Y - 8r}{4}\right) \div 7, \text{ remainder } c,$$

or

$$(Y + 2Y - 2r) \div 7, \text{ remainder } c,$$

and if you add another multiple of 7, for instance $7Y - 7r$, you get

$$(10Y - 9r) \div 7, \text{ remainder } c,$$

or

$$(10Y - 10r + r) \div 7, \text{ remainder } c,$$

or

$$[10(Y - r) + r] \div 7, \text{ remainder } c.$$

By this way the method is proved, but only for the Julian Calendar, where an intercalary day occurs every fourth year without exception.

The Gregorian Calendar was introduced in 1582 on October 15th with omission of 10 days after the Julian date of October 4th, and from that time the intercalary day is omitted in the years divisible by 100, excepting the years divisible by 400. It was omitted on 1700, 1800, 1900, but not in 1600 nor will it be in 2000. Actually the difference between the two Calendars is 13 days. In this century the Gregorian Christmas is the 12th of December of the Julian Calendar. If the Julian Christmas falls on a Thursday, the 12th of December falls on a Friday. Therefore, if you use the Julian formula to find the week-day of Christmas, you must add 1 day to get the week-day of the Gregorian Christmas. In the past century, the difference being 12 days, you have to add 2 days to the week-day of Christmas, etc., and if the correction to be added is called g, you find the week-day of the Gregorian Christmas by the formula

$$[10(Y - r) + r + g] \div 7, \text{ remainder } c.$$

If you divide the year Y by 100, you get the century C and the remainder R, or $Y = 100C + R$, and the above formula becomes

$$[1000C + 10(R - r) + r + g] \div 7, \text{ remainder } c.$$

If you take away from the dividend the multiple $1001g$, you have

$$[1000C + 10(R - r] + r - 1000g) \div 7, \text{ remainder } c,$$

or

$$[1000(C - g) + 10(R - r) + r] \div 7, \text{ remainder } c.$$

In other terms, you must subtract, in the Gregorian Calendar, from the number of the century : 1 from the year of 1900, 2 from 1800, 3 from 1700, 4 from 1582. You must subtract 1 from 1900 up to 2099, for the intercalary day is not omitted in 2000.

Therefore you find the week-day of Christmas 1958 by $18562 \div 7$, remainder 5, Thursday ; of 1858 by $16562 \div 7$, remainder 0, Saturday ; of 1758 by $14562 \div 7$, remainder 2, Monday ; of 1582 by $11802 \div 7$, remainder 0, Saturday ; of 2000 by $19000 \div 7$, remainder 2, Monday.

The Method of the Four Seasons

The method without a calendar is called the method of the four seasons. Why? In the beginning of this article we have presented the examples 18–9–1456 and 20–3–1208; it would have been possible to add 14–6–1084, the week-day being found by $14610840 \div 7$, remainder 6, Friday. The number 12 of December consists of 2 figures, but the method does not permit 2 figures. It is remarkable that one may omit 1 and put only 2 instead of 12. The week-day of 25–12–1044 is to be found by $25210440 \div 7$, remainder 3, Tuesday.

In these four examples occurred the numbers 3, 6, 9, 12 for the months of March, June, September, December, in each of which begins one of the four seasons. Now it has to be confessed that this method has a drawback: it is valid only for these four months. But the solution of this difficulty is easy: you transform the dates of other months in terms of those of the months of the four seasons, and the method is saved. You can say, the 10th April is the 41st March, the 5th May is the 66th March, the 15th July is the 45th June, the 1st January is the 32nd December of the preceding year, the 1st February is the 63rd December.

There is still one thing to be considered. The number of the year must have 4 digits; if you look for the week-day of the date 7–4–30, you must put 38–3–0030, and you find it by $38300282 \div 7$, remainder 6, Friday.

There is also a device for calculating quickly the remainder of a division by 7. You divide the dividend into groups of 3 digits, and after having found the remainder of each group, you provide the remainder alternately with positive and negative signs, beginning at the right side with the positive one. Examples:

$88611243 \div 7$, remainder 0	$89913442 \div 7$, remainder 3
$88\ 611\ 243\ \div 7$, remainder 0	$89\ 913\ 442\ \div 7$, remainder 3
$(+4\ \ -2\ \ +5) \div 7$, remainder 0	$(+5\ \ -3\ \ +1) \div 7$, remainder 3

If the remainder is negative, 7 is added to make it positive.

$$43915003 \div 7, \text{ remainder } 6$$
$$43\ 915\ 003\ \div 7, \text{ remainder } 6$$
$$(+1\ \ -5\ \ +3) \div 7, \text{ remainder } -1; \quad -1+7=6.$$

Now the method of the four seasons has to be proved. A calendar with Christmas on a Thursday has the 1st March on a Saturday, and the last day of February on a Friday. You may call this day the day 0 of March. It is 1 day after the week-day of Christmas. If the week-day of Christmas is c, the week-day w of the date D of March is to be found by the formula

$$(c+1+D) \div 7, \text{ remainder } w,$$

or

$$[10(Y-r)+r+1+D] \div 7, \text{ remainder } w,$$

for the Julian Calendar. If you add a multiple of 7, i.e. $999999D + 299999$ to the dividend, you get

$$[1000000D + 300000 + 10(Y-r)+r] \div 7, \text{ remainder } w.$$

That is the proof for March. If Christmas is on a Thursday, the day 0 of June (the last day of May) is on a Saturday, i.e. 2 week-days more. You add to the formula $(c+2+D) \div 7$, remainder w, the multiple $999999D + 599998$ and get

$$[1000000D + 600000 + 10(Y-r)+r] \div 7, \text{ remainder } w.$$

In the calendar with Christmas on a Thursday the day 0 of September and the day 0 of December are on a Sunday, 3 week-days more. You add to the formula $(c+3+D) \div 7$, remainder w, the multiples $999999D + 899997$, and $999999D + 199997$ respectively and you have the formulae

$[1000000D + 900000 + 10(Y - r) + r] \div 7$, remainder w, for September,

and $[1000000D + 200000 + 10(Y - r) + r] \div 7$, remainder w, for December.

If you want to know the week-day of a date before Christ, you have to subtract the year from 7001 ; then you treat the remainder like a year after Christ. Why have you to subtract the given year from 7001 and not from 7000? In the Julian Calendar the same week-days come back on the same dates within 28 years, a fortiori within 7000 years. But the historians drop the year 0 in the numeration of the years, putting the year -1 immediately before the year $+1$. Astronomers call the year 1 B.C. of the historians the year 0, and the year 2 B.C. the year -1.

The emperor Augustus was born in the year 63 B.C., or in the year -62 of the astronomers. $2000 - 62 = 1938$. Mussolini celebrated the anniversary of 2000 years in 1937 instead of 1938. Augustus was born on September 23rd 63 B.C. $7001 - 63 = 6938$; $23969362 \div 7$, remainder 4, Wednesday.

Cicero died 43 B.C., i.e. -42 of astronomical numeration. The teachers of many grammar-schools celebrated the anniversary in 1957 instead of 1958, which might be forgiven to philologists. But none of the mathematicians in the same schools protested, a token that mathematical chronology is not much cultivated.

A very remote date is the beginning of the Jewish Calendar on Monday the 7th October 3761 B.C. $7001 - 3761 = 3240$; $37932400 \div 7$, remainder 2.

Walchsee, Tyrol. J. M.

CORRESPONDENCE

To the Editor of the *Mathematical Gazette*

DEAR SIR,—The headmaster and the mathematics staff of this school have read with considerable interest the text of W. J. Langford's presidential address to the Mathematical Association which appeared in the October issue of the *Mathematical Gazette*.

W. J. Langford's plea for the coordination of the work done in mathematics in the later years of the primary schools with that done in the first year courses in secondary schools appeared in print at the very time when we were making our own humble efforts to " bridge the gap ".

Some months ago my headmaster, Mr. K. Allan, and I spent a morning in a local primary school watching the pupils doing English and mathematics. Shortly afterwards my headmaster decided to invite the headmasters and staffs of all the primary schools in this area to discuss with our staff the teaching of English and mathematics.

A mathematical discussion has just taken place and it was tremendously enlightening. The mathematics staff of this school are very grateful for the information they received concerning the methods and syllabuses of the primary schools, and I am sure that if others would follow this example they would be immensely pleased with the results.

Yours faithfully, A. SMITH

The Grammar School, Ashton-in-Makerfield, Lancashire

An amateur interested in number theory and cosmogony would like to correspond with a fellow enthusiast.

Please write to B. EGERTON
The Manor House
Ringwood, Hants.

To the Editor of the *Mathematical Gazette*

DEAR SIR,—Every generation must make its own discoveries. The "Stroud" system of denoting physical quantities by letters was fully ventilated in the Gazette for 1924, as your correspondent Mr Copley states. A note on this matter was Prof. C. Godfrey's last contribution to the Gazette. In it he says "the elementary convention may be cast aside at the right stage if good reason is shown. As a matter of fact, books are very far from consistent in its use. In geometry, lengths are constantly distinguished by small letters without units; in trigonometry, we find $a=1{\cdot}54''$—the very Stroud! In scholarship work no-one troubles about the elementary convention at all. . . ." After 35 years the position would appear to have remained unchanged. Prof. Lodge, in a later note, tilts at the "lamentably erroneous statement that 1 lb weight $=g$ poundals, which is found even in some text-books". Indeed it is! In the very latest text-book on Mechanics this statement is displayed, a few pages after we are told that g is the value of the acceleration due to gravity and is 32 ft/sec^2 in British units, and 980 cm/sec^2 in metric units. Now what are we to do? To be consistent, we should write 1 lb weight $=1$ lb $g=1$ lb $\times$ 32 ft./sec$^2=32$ pdl, but probably few of us would dare to do so, not because we think it necessarily difficult, but because we do not trust ourselves to be consistent about it. For myself, I am a modified Stroudian; I would preserve letters (standing for physical *quantities*) as long as possible throughout a problem, and insert numerical values in a consistent system of units at the end. But when the resistance to motion of a train is given as $v^2/200$ lb. wt. per ton, where v is in m.p.h., salvation would seem to lie in total obedience to the Stroudian Rule.

Yours etc., H. MARTYN CUNDY

P.S. Prof. Neville pointed out in 1924 that Everett introduced the "Stroud System" in 1879, whereas "forty-three years afterwards it was being put forward at the B.A. as unknown". And now after 35 years more. . . .

OBITUARY

W. C. FLETCHER

There must be many mathematical teachers in Secondary Schools who do not realise the debt that they owe to W. C. Fletcher. He was born on May 13th, 1865, and was educated at Kingswood School, Bath, and St. John's College, Cambridge. In 1886 he was 2nd wrangler, the senior wrangler being A. L. Dixon from the same school; in 1887 he was placed in division one of the first class of Part II of the Mathematical Tripos. He became a fellow of his college in 1887.

After spending nine years as a master at Bedford School, he was headmaster of Liverpool Institute from 1896 to 1904. In 1904 he was appointed to the newly created post of Chief Inspector of Secondary Schools at the Board of Education and there he stayed till his retirement in 1926. He was responsible for the organisation of the Secondary Branch of the Inspectorate and for the inauguration of the system of regular Full Inspection of Schools.

After 1926 he taught mathematics in a girls school of which his daughter was headmistress until she herself retired.

Just before he joined the Inspectorate he devised the famous trolley, which superseded Atwood's Machine in vividness and accuracy, and is still widely used in the teaching of mechanics.

In 1910 he became a member of the Mathematical Association; he gave valuable contributions to the reports on Algebra (1924), Mechanics (1930),

Arithmetic (1932), and the second geometry report (1938). In committee, whenever he spoke, his criticisms were acute and carried the greatest weight. C. O. Tuckey, chairman of the committees responsible for these reports, said that of the many able men serving on them only W. C. F. wrote drafts of which no one wished to alter a single word.

He also contributed articles to the *Gazette*: Non-Euclidean Geometry (May 1923), Lorentz Transformation (May 1925), Geometrical Congruence (about Relativity) (Jan. 1927), Napier and Logarithms (Jan. 1931), Euclid (Feb. 1938) and a most noteworthy article on English and Mathematics (March 1924) which every teacher of elementary mathematics should read and read again.

In 1939 he was appointed President of the M. A. and, owing to the war, he remained president until 1943 but his presidential address was never delivered; however, he wrote an article " Premature Abstraction " (*M. G.* May 1940) embodying the material which would have formed the basis for his address.

Here are the first few lines of the article.

"I have long been convinced that the teaching of mathematics suffers from the premature introduction of abstraction without adequate attention to the concrete facts from which the abstractions are derived."

When teaching in his daughter's girls' school, he was particularly interested in those girls who were not good at mathematics. At the end of his article " Premature Abstraction ", he wrote " My own recent teaching experience has convinced me of what I long suspected that if the pace is adjusted to the individual there are few, if any, who cannot follow the early stages of mathematical teaching without being reduced to the state of disgust or despair which has been only too common."

Of his general work at the Board of Education I will leave it for others to speak, but I must mention the circulars he issued, in particular circular No. 711 of which I will say more later. These circulars had a profound influence on teaching; circular No. 711, though not signed by him, was almost entirely his own work; of other circulars on mathematical subjects he sent round early editions to a few people for criticisms and suggestions, which he frequently used in the final form of the circular.

I was appointed by him to take part in two Full Inspections under him and I was greatly impressed by the care and thoroughness with which they were done. Twice he brought a team of Inspectors to Harrow and on each occasion within 24 hours he showed a complete grasp of the mathematical situation here.

I have mentioned above circular No. 711. This was published in March 1909; it was the first authoritative statement suggesting that no attempt should be made to teach young pupils the (so called) proofs of the early theorems in geometry, but that their truth should be led up to and made obvious by intuition and experiment; then, with these theorems assumed as a solid basis, deductive geometry should be built up.

Though it was some years before examining bodies decided not to require proofs of these early theorems, the effect of the circular was very far-reaching and led up to the reforms that have since been made in the teaching of geometry.

The circular also suggested the lines on which graphical work in algebra should be developed: it said that the aim of such work should be to lead up to functionality and the calculus instead of the watered down analytical geometry which was the feature of most of the chapters on graphs which were being hastily added to existing algebra textbooks.

In spite of his somewhat gruff manner and deep voice, which frightened some teachers, he was a most kindly and lovable person. He was a man of

wonderful physical vigour: during the last few years of his life his eyes began to fail and he ceased to be able to read and write, but he was still very active in mind and body: after he was 90 he told me that he was busy sawing a tree into planks, though he regretted that he could no longer play his 'cello. And only a fortnight before his death he dictated a letter to me in which he said "I have been keeping very well: I get out of doors quite freely, though it will be different when the snow comes."

Last year he was made an honorary member of the Mathematical Association; this gave him very great pleasure.

A. W. SIDDONS

A brief notice† enumerating the successive steps in the academic and professional career of W. C. Fletcher would in one sense be appropriate to the subject; for Fletcher was a man of few words.

Yet a bold statement of facts could hardly do justice to an outstanding personality who made a noteworthy contribution to the standards of Grammar School education that we have in England to-day.

The Secondary Schools first recognised as such in Edwardian times by the recently created Board of Education had originated in these different ways. Some were "Board Schools" intended under (or in spite of) the old Elementary Code; some had been recognised as Organised Service Schools by the South Kensington Department; others were Endowed Grammar Schools not assisted by any government department up to that time. To these were added, in due course a large number of new foundations by Local Education Authorities under the Act of 1902. In assembling a staff for the care and oversight of this rather diverse group, Fletcher drew some from Whitehall—*i.e.* H.M. Inspectors of Schools under the Elementary Code. He took half of the "Twelve Apostles" of South Kensington. These were joined by others recruited like himself from "Public" and Grammar School staffs.

Under his leadership we soon became a band of brothers. The Full Inspection, which he instituted, was a far-reaching survey of school teaching and organisation taking cognizance of every aspect of school life. District Inspectors were combined in teams and joined by colleagues from Whitehall.

He was none the less a man of wide culture and deep religious conviction. In fact, he very warmly welcomed from the outset any invitation to inspect Religious Instruction as an element of the curriculum. He was as much at home in a Latin lesson as in a mathematical one. In music he was a student of Bach. He had a fine appreciation of English poetry but no small talk, and it may not be amiss to put on record a view he experienced in connexion with the writing of essays; "When a child has nothing to say, he should be urged to silence and not to speech". JAMES STRACHAN

† Reprinted, with permission, from "The Times".

PLAITED POLYHEDRA†

BY A. R. PARGETER

1. In Cundy and Rollett's invaluable book on *Mathematical Models* [1], the authors begin their Chapter on Polyhedra by remarking that "The most suitable, and in many ways the most attractive, subject for an experiment in the construction of mathematical models is a set of polyhedra." Various methods are in general use to produce finished models of polyhedra for the showcase, and similar methods form the basis of those considered suitable for constructing the simpler polyhedra in the classroom. That most commonly adopted is to draw the net of the required solid on a sheet of cardboard, allow for tabs as necessary, cut it out, score the creases half-through, fold up, and stick the tabs with some suitable quick-drying cement. For class-room use the major drawback to model making on these lines is the time-consuming and potentially messy process of sticking; and it is the object of this article to develop a method which dispenses with the use of paste or cement altogether. In fact I shall show how, using paper and scissors only, firm, neat models can be made, which in the case of the simpler solids could easily be produced in the class-room, and which in the case of the more complex polyhedra reveal by their construction in a striking manner some of the geometrical relationships of these solids.

2. My attention was first drawn to this idea by a reference in *Multi-Sensory Aids in the Teaching of Mathematics* [2] to an out-of-print book by John Gorham, *Plaited Crystal Models* [3], published in 1888. The article in *Multi-Sensory Aids* is too condensed to be very illuminating, and is illustrated by a single and not very well chosen example, and I did not attempt to pursue the matter further until I managed to lay hands on a copy of Gorham's book, when my interest was fully aroused. John Gorham was a medical man with an interest in crystallography. He states in his Preface: "It is now some forty years since I had the honour of demonstrating before the Royal Society in London *A System for the Construction of Crystal Models Projected on Plane Surfaces*. These figures folded into the required form, and subsided into a level at pleasure—they were easily moulded into shape by bringing their edges into apposition with the fingers, and were as easily transferred from place to place when flattened in a portfolio . . ." He naturally confines his attention to those forms in which a crystallographer would be interested, namely the cube, regular tetrahedron and regular octahedron, numerous irregular versions of these (including what he terms rhombohedra), triangular and hexagonal prisms (the latter with and without pyramidal ends), and the rhombic dodecahedron. Not all of these solids are of equal interest to the mathematician, for whom there are obvious gaps, in particular the regular dodecahedron and icosahedron. I therefore set out to see whether Gorham's methods could be applied to these solids; and, succeeding, to extend them to the construction of Archimedean, dual, and stellated polyhedra. The time at my disposal has not permitted me to carry out the construction of more than a small selection of all the possible models, but in many cases it has been sufficient to observe that an effective construction *is* possible without being obliged to carry it out to verify the assertion: I shall show later how once a polyhedron has been constructed by a "compound plait", a simple modification at once leads to the construction of a range of related (though possibly quite different) polyhedra, and so it is only necessary to make one member of the set to have available the procedure for constructing the others. I shall

† A lecture delivered at the Annual Meeting of the Mathematical Association in April 1959.

show that it is theoretically possible to plait *any* polyhedron; but in some cases the method may be impracticable owing to too great complexity. Gorham does not appear to have made any attempt to generalize his methods, or to develop the underlying theory—in fact, as one extract quoted in *Multi-Sensory Aids* shows, his approach was primarily experimental; and he remained content—so far as one can judge from his book—with those models in which he (as a crystallographer) was interested. I shall show later how methods for plaiting any desired solid can be worked out systematically.

3. It is fitting to preface the descriptive part of this article with Gorham's own measured words: " It is a property inherent in the ordinary plait that its constituent parts shall cohere compactly without adventitious aid. This involves definite arrangement; and if this arrangement is studiously followed in the construction of models, their faces will in like manner become coherent, and, when plaited, will assume a solid form, and maintain it without extraneous assistance."

4. In proceeding to describe in detail the method of making plaited models, I shall use the term " net " to designate the net of a polyhedron as prepared for plaiting, and not the net as ordinarily used for a conventional model. It should also be noted that as there is some lack of uniformity in the naming of a few polyhedra—especially the Kepler-Poinsot star polyhedra—I shall adhere to those names adopted in *Mathematical Models*. It will in fact be assumed that the reader of this article is already acquainted with the names and forms of the Pythagorean, Kepler-Poinsot, Archimedean, dual, and stellated polyhedra: it would take too long to enumerate them here, and the information can be found, in varying degrees of completeness, in several well-known books (e.g. Rouse Ball's *Mathematical Recreations and Essays* [4], but especially *Mathematical Models* already referred to.)

5. In setting out to make the actual models the following points must be noted:

5.1. In the nets as here depicted, *thick* lines should be cut, *thin* lines should be creased; a short stroke across a line indicates a reversed crease; *dotted* lines are sometimes inserted to indicate structure or to facilitate description.

5.2. On account of the amount of overlapping involved, *firm paper* (*e.g. cartridge paper*) *is preferable to cardboard* (although very thin cardboard could be used for large models).

5.3. Assuming that the net has been accurately drawn and cut out, *firm* and *accurate* creasing of the folds is essential for the production of a neat model. Paper cannot safely be scored with a penknife (as would be done in the case of a cardboard model), but accurate creasing can nevertheless be greatly facilitated by indenting the lines to be creased with a fairly sharp paper-knife.

5.4. It is assumed that the reader knows how to form an ordinary 3-strand plait, such as a small girl uses for her pigtails. Sailors have invented a variety of interesting plaits, including " solid " ones; but we are concerned only with " flat " plaits, being a natural extension of the 3-strand plait to any number of strands. The basic idea is that the strands pass over and under each other in strictly regular order, alternate strands moving the same way. In a true flat plait the strands are turned back each time they reach one or other of the edges of the plait, but in plaiting over a closed surface, such as a polyhedron, the strands maintain their own directions until they return upon themselves, after which they would continue to encircle the surface indefinitely. In designing a plaited model the strands are allowed to overlap their own beginnings just sufficiently to provide adequate " tuck-ins " (see § 5.5).

5.5. Due attention having been paid to §§ 5.3 and 5.4, folding up the model is almost automatic, provided the *first* overlap is correctly made. To this end the section marked O on one strip must be folded over the section marked U on another, the words " over " and " under " being interpreted on the under-

standing that the model is being viewed from the outside, i.e. that it is being so held during the course of construction that the outside is uppermost; the over-and-under action of plaiting will then bring each next section to be folded in position as it is needed, and finally one or more ends will remain to be tucked in, which will render the model firm and self-supporting. The nets here shown have been so designed that once the polyhedron is complete, the minimum necessary amount of overlapping is allowed to permit of a finished model in which the places at which the final tucks are made cannot be determined by visual inspection—in fact a neatly made model can be very difficult to undo! If the first fold is incorrectly made (i.e. if the roles of O and U are interchanged), this will result in the ends of the strips having nowhere to tuck in. The reader wishing to master the art of plaiting polyhedra is advised to start with the simpler ones, where the process will be easy to follow, and work up to the harder ones; the more complex models require a knack which, however, once acquired by practice, enables them to be folded up with surprising rapidity, even when 6 or 8 strands are being simultaneously manipulated. Stellated polyhedra are generally rather tricky, but feasible nevertheless; temporary paper clips are helpful.

5.6. Since the surface of the polyhedron is formed of overlapping strips, and allowance has to be made for final tucking in, the number of faces in the net exceeds twice the number of faces of the polyhedron; moreover the sequence of parts cannot be much varied. Thus only small models can be made with moderate sized pieces of paper; but the method of § 5.8 can be used to overome this.

5.7. Models involving only equilateral or isosceles 120° triangles can most conveniently be made with the help of isometric graph paper, which is ruled into small equilateral triangles (as obtainable, e.g., in the " Chartwell " series, no. 4801—triangles approximately 5 to the inch, effective area of sheet $10'' \times 16''$). The "Chartwell" paper, which is plain on the reverse side, would in fact be excellent for making models, but as it is rather expensive for cutting up wholesale it is best to use it to determine the vertices of the net or strip, which can then be pricked through on to plain paper.

5.8. A convenient method of preparing the net is to make an accurate construction of a single strip and then prick it through as many times as required—in fact time may be saved by cutting out several together, but to attempt too many at one cutting may result in inaccuracy (a useful dodge, which will reduce the risk, is to staple the sheets together before cutting, being generous with the staples). The strips are then joined together to make the complete net with the aid of small pieces of adhesive paper, and trimmed to the correct lengths as determined from the diagram of the net. This may appear to violate the principle of " no sticking ", but it is very quickly and neatly done—the model still being in the flat—and the joins, being inside the finished model, will be unseen. (In theory the strips need not be joined at all, as they would retain their positions in the finished model, but the practical advantages of fixing them in their relative positions at the outset are obvious. Alternatively, the strips could be fastened together by means of slide-on paper clips, which can be removed just before the model is completed; for this purpose the strips must be made long enough to overlap at the outset, rather than joined by edges as they are if stuck. Models which are likely to be undone and refolded at all frequently are however not conveniently made this way, and the firmness of larger models is affected.) This use of separate strips is moreover the only way of making the nets of some polyhedra in which the strips, being curved, cannot be arranged even in the plane net so as not to overlap. It also enables (as mentioned in § 5.6 above) larger models to be made than can be obtained when the whole net has to occupy a single sheet of paper; and if, on these lines, we use different coloured papers for the

strips, the results are extremely attractive—and would make fine decorations for the Christmas tree! (I can strongly recommend making in this way the group of models based on 15 (see below), especially the great dodecahedron, 20. It should be noted, however, that although the colouring does help to

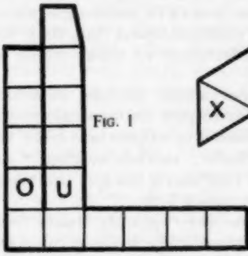


FIG. 1

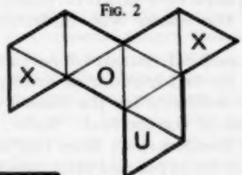


FIG. 2

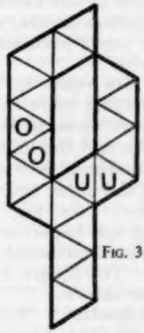


FIG. 3

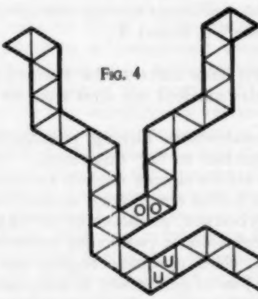


FIG. 4

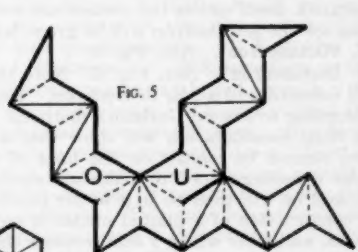


FIG. 5

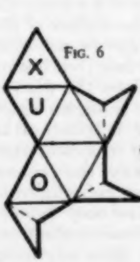


FIG. 6

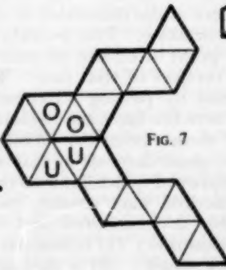


FIG. 7

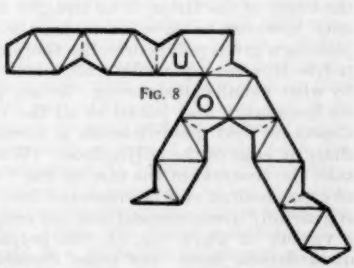


FIG. 8

bring out the inter-relationships of such a group of polyhedra when viewed in accordance with the ideas of this article, it has no geometrical significance in relation to the individual polyhedron—e.g., it would be more logical to have a great dodecahedron with all five parts of any one plane face rendered in the

same colour, whereas when plaited the 12 faces display each of the 6 combinations of 5 different colours chosen out of the 6 (as there are 6 strips), in inverse rotation on opposite faces.

5.9. All construction lines, if any, and joins (if any—as in § 5.8) should be kept on the same side of the net, and the creases can then be made so that this is the inside of the finished model. Several other points of technique which help to increase accuracy and save time could be particularised, but they will doubtless occur naturally to the reader who undertakes to make many of these models.

6. We now come to the practical details of particular models, of which space precludes more than a limited number of examples being dealt with ; I try, nevertheless, to include sufficient for the reader to appreciate both the possibilities and the limitations of the method. Note : models marked * are taken (sometimes with slight modification) from Gorham's book ; the significance of the index figures will be explained later (see § 7.2).

1. ***Cube**[3]. Net, Fig. 1. A small model can be very quickly made from ordinary graph paper, as the net can be cut out immediately without even the use of a pencil (although in the classroom it would be advisable to outline the net first). The square A that is finally tucked in goes more easily if slightly tapered, as shown.

2. ***Tetrahedron**[2]. Net, Fig. 2. The triangles marked × could be dispensed with, but add to the rigidity of the model. The last triangle is a little inclined to untuck itself unless the creases are very firm. (Firmer though less simple plaits for the tetrahedron will be given later—see § 8, **5** and **6**.)

3. ***Octahedron**[1]. Net, Fig. 3.

4. **Icosahedron**[1]. Net, Fig. 4. Note that two ends have to be tucked in, each consisting of a pair of triangles. This model marked my first success at attempting to extend Gorham's methods.

A little consideration will show that a dodecahedron, having pentagonal faces, cannot be plaited on the lines of the models so far described. The reader is recommended to make and study the above simple models for himself, and he will then be in a better position to follow the theory underlying the construction of a plaited model of any polyhedron, an account of which follows, and after which I shall resume the description of particular models.

7. General Theory. Where two strips cross, their common region has 4 vertices, and so is a quadrilateral, for there is no loss of generality in supposing the edges of the strips to be straight between the vertices. The quadrilateral may, however, be skew, i.e. folded about a diagonal. The basis of a scheme for plaiting a given polyhedron lies therefore in the dissection of the surface of the polyhedron into quadrilaterals, plane or skew. This can always be achieved by what I shall call *sectoring* : let any point be chosen on each face (but not on its boundary), and joined to all the vertices of that face. Then the desired dissection into quadrilaterals is formed by pairing off those triangles which share an edge of the polyhedron. (Where the faces are regular, it is natural to take the centroid of the face as the " chosen point ".) Whilst it is clear that any polyhedron can be dissected into quadrilaterals in this way, the process is generally uneconomical and the number of quadrilaterals can be reduced in a variety of ways, e.g. (1) the polyhedron may already have only (plane) quadrilateral faces—the cube, rhombic dodecahedron and triacontahedron, and trapezoidal icositetrahedron for example ; (2) triangular faces adjoining each other may be paired off as they stand ; (3) a face adjoined *only* by triangular faces (e.g. a square face of a cuboctahedron) if sectored yields quadrilaterals if the sectors are paired with the complete adjoining triangles; (4) a polygon may be dissected into quadrilaterals and triangles in various ways, and the triangles can be paired off with adjoining triangular faces or sectors.

Having dissected the surface of the polyhedron into quadrilaterals in some way, we now arrange these quadrilaterals into sequences in which the successive quadrilaterals of a sequence are joined by opposite edges. Such a sequence of quadrilaterals forms one of the strips of which the plait is composed.

7.1. The following facts about a sequence so constructed can now be remarked : (*a*) The sequence may cross itself. (This presents no difficulty, it will be found, when the actual scheme for plaiting is developed.) Or, to put it another way, every quadrilateral comes twice, either once each in different sequences or both times in the same sequence. (*b*) Every sequence is closed. For the number of quadrilaterals is finite, and so we are bound to return to the starting point even if we have to cross every quadrilateral in both directions in the process—i.e., there may be only one sequence (and this frequently happens, e.g. in an icosahedron with a suitable pairing of the faces (**4**)). (*c*) The number of quadrilaterals in a sequence is even. If the sequence is unique, it must cross itself everywhere, and the result is obvious. If not, its edges form the polygonal boundaries of a number of simply-connected regions of the surface. If the number of quadrilaterals in the sequence is odd, some (at least 2, in fact) of these polygons must have an odd number of sides. But a polygon with an odd number of sides cannot be dissected into quadrilaterals (proved inductively by observing that if we subtract from a polygon of n sides a quadrilateral which shares 1, 2 or 3 sides with the polygon then the remaining polygon has $n+2$, n or $n-2$ sides respectively, so that n, if odd, can never be reduced to 4). This contradicts the assumption that the surface of the polyhedron is completely dissected into quadrilaterals. (*d*) Following the edge of a sequence, label the vertices A, B alternately. It follows from (*c*) that every vertex of the system of quadrilaterals can be so labelled, in such a way that no two consecutive vertices have the same letter. It is now evident that if we think of the sequences of quadrilaterals as strips (of paper), they can be plaited by following the rule that at each A vertex we step " down " from one strip to the one that passes under it if we trace a small contour round the vertex in an anticlockwise sense, whilst at each B vertex the same thing happens if we trace such a contour clockwise. This completes the demonstration that any polyhedron can be plaited ; and it follows from § 7 that a given polyhedron may be plaited in more than one way.

7.2. If the number of closed sequences on the surface of the polyhedron, i.e. the number of strips needed to form the plait, be n, I call the resulting model an n-plait. It would appear at first sight impossible to plait with only 1 or 2 strands ; this of course is true of a linear (pigtail) plait, but on a closed surface a strand may cross itself repeatedly, and 1-plaits and 2-plaits exist. It is, however, not practicable to *execute* a plait with a single strand or two strands, as this would necessitate forming part of the plait loosely and then threading the rest in and out like a shoelace ; to carry out the action of plaiting it is necessary to manipulate simultaneously at least 3 strands (with the exception of certain small models, e.g. octahedron[2] (**7**), triangular prism[1] (**33**)), and so in designing the net for (say) a 1-plait it is necessary to cut the strip into at least 3 parts and join them together at intersections—this is what has been done, e.g., for the icosahedron[1] (**4**). Again, the rhombicuboctahedron and snub cube can most simply be formed as a 3-plait in which one strip is (this is unusual) of different structure from the other two, and this strip is so much longer than the others that it has been broken into three pieces for plaiting so that the net appears to be that of a 5-plait (see **29**). The index figures attached to the names of models throughout this article indicate the number of strips of which the model is composed. It should also be made clear that I use the word *strand* rather than strip when referring tothe process of plaiting the model rather than the theory of its construction ; thus the

icosahedron[1] is a plait formed of one strip, but so arranged that in plaiting one manipulates 3 strands. The multiplicity of a plait is clearly brought out by cutting the different strips from different coloured papers; besides being instructive, the results are (as has already been mentioned) extremely attractive.

8. We are now in a position to understand the construction of some further models. But first, returning to those already described, and looking at them in the light of the preceding paragraphs: the construction of the cube is obvious, while the tetrahedron, octahedron, and icosahedron are readily dissected into quadrilaterals by pairing the faces. There are two ways of doing this for the octahedron, of which the alternative pairing gives a 2-plait described below (7); and the faces of the icosahedron have been so paired as to make it structurally equivalent to a pentagonal trapezohedron. To proceed:

5. ***Tetrahedron**[3]. Net, Fig. 5. This is a very firm, neat model, obtained by sectoring all 4 faces. (In this, as in some other small models, the end quadrilaterals of the strands have to be truncated as shown or they would be impossible to tuck in.)

6. **Tetrahedron**[1]. Net, Fig. 6. Only one face is sectored. This is a very useful model, as it is much firmer than 2 but simpler than 5. (The triangle marked × is not strictly part of the plait, but is added for firmness. Similar additions will be found in some of the other small models.)

7. **Octahedron**[2]. Net, Fig. 7. A different pairing of the faces from that used in 3 leads to this neat 2-plait, in which the two strands are simply "twisted" together to make the model. This is, I consider, easier to make than Gorham's octahedron (3), but care must be taken that it does not fold itself into a triangular dipyramid (see 27).

8. **Octahedron**[2]. Net, Fig. 8. Two opposite faces sectored, giving a very firm, neat, yet fairly simple model.

9. ***Octahedron**[4]. Net, Fig. 9. In this and the next model, all faces are sectored.

10. ***Cube**[4]. Net, Fig. 10. See note to 9.

11. ***Rhombic dodecahedron**[4]. Net, Fig. 11.

8.1. The relationship between the last three models merits careful study. In 10 the faces of the cube are divided into right-angled isosceles triangles, and the basic quadrilaterals of the strips are squares folded each about a diagonal, but not along their common edges. Now suppose that these squares are replaced by congruent rhombuses with acute angles at those vertices corresponding to the uncreased diagonals, and that additional creases are made along their common edges; then on plaiting the model, a pyramid will be raised on each face of the cube. (In particular, if we take the above mentioned acute angles to be 83° 37′ we shall obtain a

12. **Tetrakis hexahedron**[4].

But if the additional creases are reversed, the faces of the cube will acquire pyramidal "dimples".) Suppose now that we raise pyramids of such a height that the adjoining faces of two adjacent pyramids lie in the same plane; we then have a solid with as many rhombic faces as the cube has edges, namely a rhombic dodecahedron: the diagonal creases can now be dispensed with, and we have model 11. (The acute angle of the rhombus for this purpose is 70° 32′, but it is perhaps easier to make use of the fact that its diagonals are in the ratio $\sqrt{2} : 1$.) In the same way, starting with an octahedron plaited as 9, on replacing the 120° angles of the rhombuses of which the strips are composed by 117° 14′ and creasing along the common edges we obtain a

13. **Triakis octahedron**[4];

and by making them 109° 28′ (the supplement, of course, of the 70° 32′ mentioned above) and omitting the diagonal creases we obtain again the rhombic dodecahedron 11. Further, by making the same angles 60° and

reversing the diagonal creases a tetrahedron will be raised on each face of the basic octahedron and we obtain a

14. Stella octangula[4]. Net, Fig. 12. (This is rather tricky to fold up.) Thus we see that all the models **9** to **14** are formed from basically the same plait, and the similarity of the nets is evident. I use the term *compound plait* to designate a plait formed by sectoring all the faces of a polyhedron, e.g. **5**,

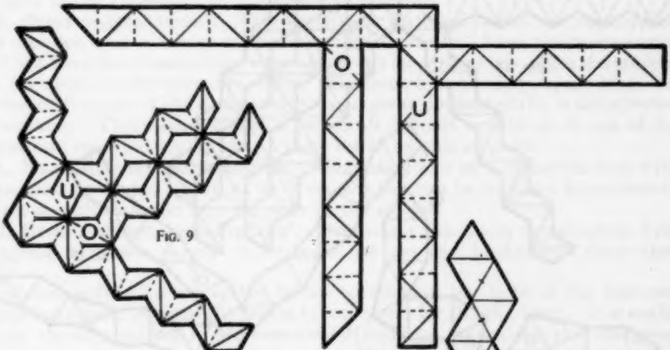


Fig. 9

Fig. 10

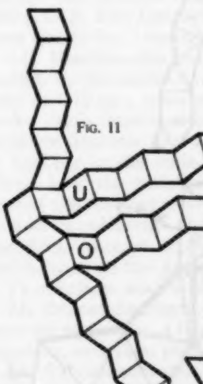


Fig. 11

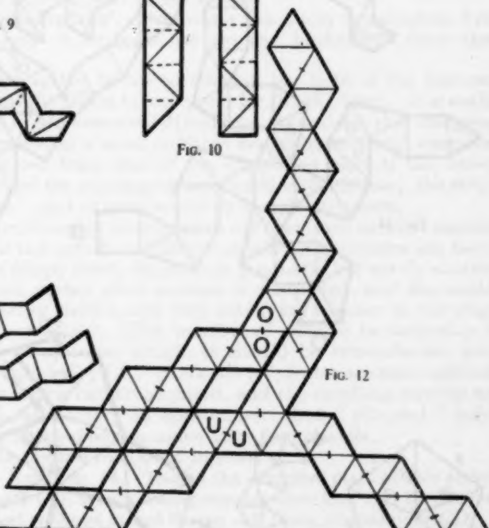


Fig. 12

+ indicates reverse fold

9 and **10**. It is now clear that we have available a system whereby a range of related polyhedra can be formed from one basic net by altering the angles of the constituent quadrilaterals and adjusting the position and " sense " of the creases in conformity. (In particular it should be remarked that the compound plaits of any convex polyhedron and its dual differ only in (*a*) the angles of the quadrilaterals and (*b*) the transference of the crease from one diagonal

to the other.) In this way, an interesting group of models can be based upon the compound plait of the icosahedron, as follows :

15. Icosahedron[6]. Net, Fig. 13. The basic quadrilaterals are 60°, 120° rhombuses, creased along the long diagonals.

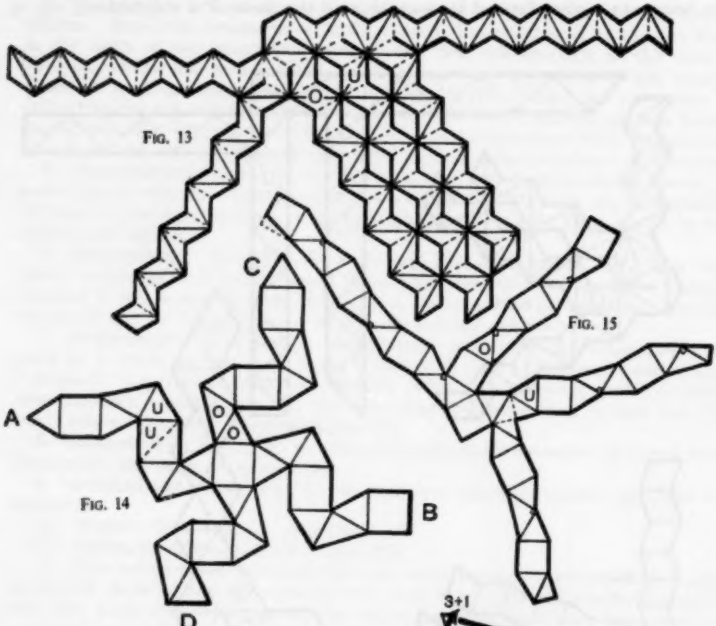
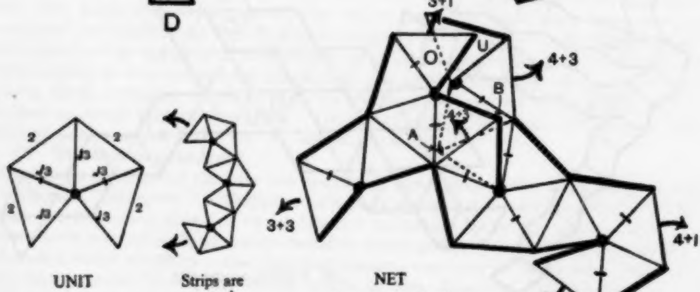


FIG. 13

FIG. 14

FIG. 15



UNIT

Strips are curved

NET

Arrows show further number of whole units + extra triangles at end

FIG. 16

16. Triakis icosahedron[6]. In 15, replace the obtuse angles by 119° 3′, and crease additionally along the common edges.

17. Rhombic triacontahedron[6]. In 15, make the obtuse angles 116° 34′ and crease along the common edges only.

18. Pentakis dodecahedron[6]. In **15**, make the obtuse angles 111° 24′ and crease along the common edges and the short diagonals.

19. Dodecahedron[6]. In **15**, make the obtuse angles 108° and crease along the short diagonals only. This is the compound plait of a dodecahedron, and its relationship to that of the icosahedron via the rhombic triacontahedron is the same as that of the cube to the octahedron via the rhombic dodecahedron. No substantially simpler plait for the dodecahedron appears to be possible, which is a pity as it means that this only of the 5 Platonic solids cannot easily be made in the classroom by the present method.

20. Great dodecahedron[6]. This solid may be regarded as an icosahedron with dimples, and it is easy to discover that its net will have the same angles as **19** but that the creases must be made on the *long* diagonals of the rhombuses and, reversed, on the common edges. The complete identity, apart from the creases, of the nets of the dodecahedron and great dodecahedron is unexpected and striking. This model, which is not at all difficult to fold up, is one of the firmest and most satisfying that I have made by this method.

21. Small stellated dodecahedron[6]. Regarding this as a dodecahedron with pentagonal pyramids raised on its faces, the net can be deduced from that of **19** in a way that should now be clear to the reader.

22. Great stellated dodecahedron[6]. Regarding this as an icosahedron with triangular pyramids raised on its faces, its net can be deduced from that of **15**.

Stellated polyhedra are rather tricky to fold up, but both of the last two models have been successfully made by my pupil Mr. R. W. Bray. It is easily seen (v. the illustrations in *Mathematical Models*, pp. 84 and 90) that the great icosahedron may be regarded as a small stellated dodecahedron with dimples ; its net can thus be deduced from that of the compound plait of the latter. Mr. J. C. F. Fair has tackled the plaiting of this model with success ; the strips turn out to be " straight " and of comparatively simple structure.

8.2. Readers who are sufficiently interested to try their own hand at making models by the method of this article will find that convex polyhedra are fairly easy to fold up ; after a floppy start, the strands gradually but surely shorten under one's fingers as one vertex after another is completed, and the model acquires firmness in a rather sudden and very satisfying manner as the stage of tucking in the ends is reached. (The number of ends to be tucked in is about half the number of strands—which, it should be remembered, may exceed the number of strips : cf. § 7.2.) Stellated polyhedra are more difficult to fold up, although the knack can be acquired, and the resulting models are less firm—though quite satisfactory in appearance ; but " dimpled " polyhedra, such as the great dodecahedron, make very firm models.

To continue with some further models of interest :

23. Cuboctahedron[2]. Net, Fig. 14. This is the simplest plait of this solid ; the two strips are *AB* and *CD*, which, being almost identical, can be cut out together, and then plaited without initial fixing, *AB* being placed over *CD*.

24. Cuboctahedron[4]. Net, Fig. 15. This is a rather neater model than **23.**

25. Cuboctahedron[6]. This is the compound plait of the solid, and it is hardly worth making on its own account, but from it can be deduced the net of the compound plait of its dual, the rhombic dodecahedron (or of course this can be worked out independently), and from either that of the

26. Stellated rhombic dodecahedron[6]. Net, Fig. 16. I have made a successful model of this. When folding, note that at the start the corners marked *A* and *B* are brought together to form an outward-pointing vertex.

Deltahedra are polyhedra all of whose faces are equilateral triangles (see *Mathematical Models*, pp. 72 and 135f.). They include, of course, some of the regular polyhedra, but I have also made several of the others, both convex and non-convex ; the irregular ones need folding with care, as they have a dis-

concerting tendency to fold up into the wrong deltahedron. A very neat model is the

27. Triangular dipyramid[1]. Net, Fig. 17. This net is a shortened version of that of the octahedron[2] (7).

Another example is the

28. Dodecadeltahedron[3]. Net, Fig. 18.

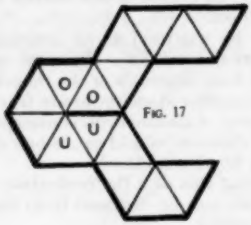


FIG. 17

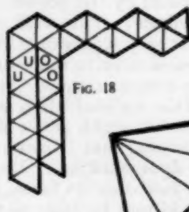


FIG. 18

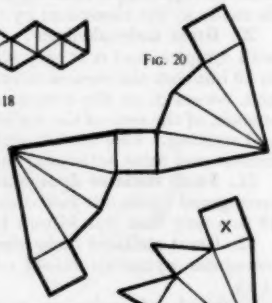


FIG. 20

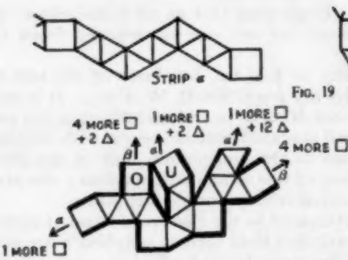


FIG. 19

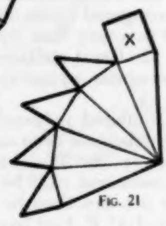


FIG. 21

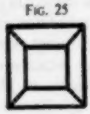


FIG. 25

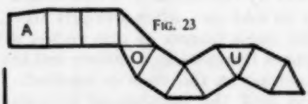


FIG. 23

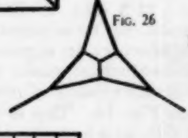


FIG. 26

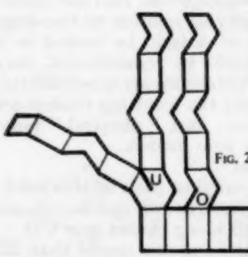


FIG. 22

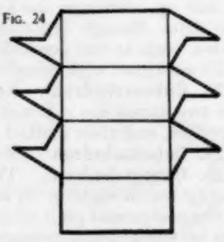


FIG. 24

If the " cube " faces of a rhombicuboctahedron are left alone, and all the other square faces divided by diagonals (which must be suitably chosen), then on replacing all the right-angled isosceles triangles so obtained by equilateral triangles we get a snub cube. The nets of these two solids will therefore be closely related. The snub cube can be dissected into quadrilaterals by suitable pairing of the triangles (not, of course, that corresponding to the rhombi-

cuboctahedron, which would leave odd ones over), and the resulting sequences give two short strips and one long one of different structure, so that it is in fact a 3-plait; but the net has the appearance of a 5-plait as the long strip has to be broken into 3 pieces to make the model workable. I illustrate the snub cube, as being the more interesting of the two solids. The enantiomorph should present no difficulty. Alternatively, undo the one given, reverse the creases, and refold—plaited models can be turned inside out in *three* dimensions!

29. Snub cube[3]. Net, Fig. 19.

8.3. There is of course no need for the polyhedra to have regular faces for them to be successfully plaited: in fact a large part of Gorham's book is devoted to forms which the geometer would regard as irregular but which occur in the study of crystals. The reader of this article should have no difficulty in modifying the nets of (e.g.) the tetrahedron and cube to be able to plait scalene tetrahedra, cuboids and parallelepipeds.

There are a few other solids of frequent occurrence whose nets I now give, some of the forms (those with no index) being *ad hoc* constructions not strictly falling under the general theory.

30. Square pyramid[1]. Net, Fig. 20. This is of interest as being a 1-plait in which the single strand can be folded up as it stands. The next model is however simpler and neater.

31. Square pyramid. Net, Fig. 21. The square marked × is added for firmness. A resemblance will be noted to the tetrahedron plaited as in **6.** The idea can be successfully extended to pyramids of any number of sides.

32. *Hexagonal prism[4]. Net, Fig. 22. A hexagonal prism with the ends divided into three rhombuses is structurally equivalent to a rhombic dodecahedron (with some creases omitted), but I give the net here separately as a model of this solid is likely to be in demand. To make a bee's cell, replace one end of the prism by a set of 3 rhombic dodecahedron faces, and modify the rectangular faces accordingly (using the same angles as for the rhombuses).

33. Triangular prism[1]. Net, Fig. 23. Another 1-plait that can be folded up as it stands. The part A should strictly be a triangle, but a rectangle (tapered for ease in tucking-in) is less liable to come undone.

34. Triangular prism. Net, Fig. 24. This is an *ad hoc* construction which makes a neat model.

Gorham's own version of the triangular prism I do not give, as it is cumbersome compared with either of the above.

8.4. Amongst other models which I have made by this method are the great dodecadodecahedron, and the (space-filling) truncated cuboctahedron. In *Multi-Sensory Aids* the expression *intraverted polyhedron* is used to describe the pseudo-solid obtained by joining the edges of a polyhedron by means of planes to its centre, and removing the original faces—thus each such face is replaced by a pyramidal dimple, all the pyramids having a common vertex. An *extraverted polyhedron* is the same solid with the pyramids reversed on to the outside of the figure. Using this terminology, it is well known that an extraverted cube is a rhombic dodecahedron; hence an intraverted cube is easily plaited from the net of the rhombic dodecahedron by creasing along the short diagonals and reversing the original creases. To make an intraverted tetrahedron, take the net of the tetrahedron[3] (5) and make the acute angle of the rhombuses 70° 30′.

9. The designing of a net for plaiting a desired polyhedron—if not deducible from one already known by methods such as those described in § 8.1—is greatly facilitated by the use of a " distorted " or Schlegel diagram. There are two versions of these: (*a*) in which one face of the polyhedron is removed and the remainder of the surface is shrunk and flattened so as to lie within the boundary of that face (which it is now convenient to think of as represented by the rest of the plane), and (*b*) in which the polyhedron is similarly flattened

but one vertex is projected to infinity. Without troubling to make these descriptions more precise, the processes will become clear on referring to Figs. 25 and 26, which show the results of (*a*) and (*b*) for a cube.

If the Schlegel diagram is sketched in pencil, the dissection into quadrilaterals can be done in ink, and then the sequences of quadrilaterals can be followed by means of differently coloured lines, and the actual forms of the strips then worked out. It should be clear that there is no choice in the matter of the sequences—they are determined completely by the dissection into quadrilaterals, but this, as already pointed out, can be done in numerous ways.

The simplest procedure, of course, is to trace out the strips on an actual (not necessarily plaited) model, if one is available!

10. In one of the earlier attempts at experiment in the teaching of geometry to beginners, by G. C. and W. H. Young [5], the authors encourage throughout the making of models, which they have designed so that they can be folded up and secured by means of tucked-in or interlocking flaps without sticking. The net so prepared they call a " flat pattern ", and consists of the conventional net with the addition of sufficient extra faces to enable a self-supporting model to be formed on similar lines to those of this article. For so simple a solid as the regular tetrahedron it is not surprising to find that their flat pattern is identical with the net for plaiting as given by Gorham (which is **2** above without the triangles marked ×), but for the cube and all other models the positioning of the additional faces in the flat pattern, although sufficient for the purpose, is otherwise quite arbitrary, and the models cannot be readily folded up without a knowledge of the exact order of interlocking of the faces. To this end the faces in the flat pattern are numbered and lettered systematically: Gorham does the same thing, but in the case of his models it is unnecessary, with the exception of the first fold, as the beauty of plaiting is that provided this is made correctly the remainder of the process of folding, however large the model, is automatic. The complications of plaiting (say) a dodecahedron do not arise in the Youngs' book, as they do not go beyond the octahedron and some simple irregular solids (including Bimbo's Lozenge!—a deltahedron in the form of a pentagonal dipyramid, which incidentally they incorrectly describe as being formed from five regular tetrahedra. This can be made by extending the net of **7** in the same way that the latter can be obtained by extending the net of **27**.) The Youngs' models are less firm than plaited ones, as there are more " ends ". An interesting feature of the flat patterns, however, is that they are all obtained by folding a plain sheet of paper; no construction lines are used, the only apparatus needed being a sheet of paper and a pair of scissors. (I wonder how many readers could fold an equilateral triangle?) This idea can quite well be applied to some of the simpler plaited models; it requires skill, and care not to introduce unwanted creases in plane faces, but can save time when mastered.

11. It is surprising that the idea of plaiting a cube, tetrahedron, and octahedron by the simplest method (**1, 2, 7**) is not more generally known, and possibly there have been others besides Gorham and the Youngs who have struck out on these lines; but to Gorham must go the credit of making these solids by what I have termed a compound plait. I do not know how he hit upon this method, although presumably it arose from the nature of the crystal structures in which he was interested; but it was a brilliant and fruitful inspiration, without which this article would not have come to be written.

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WHAT IS × ?

BY H. MARTYN CUNDY

Would you indicate $3+3+3+3$ as 4×3 or as 3×4? Most colleagues and boys that I have asked plump for the former. 4×3 means to them "four threes" or "four times three", which is emphatically not the meaning "four multiplied by three". Clearly "four times three" is the same as "three multiplied by four"; (I do not mean that they are *equal*, I mean that the expressions are two ways of saying the same thing). If we indicate this common meaning by the symbols 4×3, we are in fact doing the opposite of what we do in the case of the other three arithmetical processes. $4+3$ means (to most people) "take four and add to it three", or "four increased by three"; $4-3$ means "take four and subtract three from it", or "four diminished by three"; so $4\div 3$ means "take four and divide it by three". Obviously, then, 4×3 must mean "take four, and multiply it by three"; *i.e.* "four taken three times" or *three fours*.

But inevitably it soon gets turned round. We write $a+a+a=3a$, not $a\times 3$, as we should. $3a$ means "three a's"—*i.e.* "a multiplied by 3". In fact the rule for algebra is here, and here only it seems, the opposite of that for arithmetic. In algebra we habitually put the multiplier *before* the multiplicand: witness our preference for $a(b+c)$ over $(b+c)a$. Most boys will write $x(2x+3y)-2y(2x+3y)=(x-2y)(2x+3y)$ where most teachers would probably prefer $(2x+3y)(x-2y)$. But perhaps the boys are not being quite so perverse after all! Is it not odd that in algebra where above all × ought to be read "multiplied by", it (or .) is habitually construed as "times" and often (at least by boys) so read?

The algebraic rule persists with operators. In dealing with expressions such as $x\,.\,\dfrac{d}{dx}\,.\,y$ and $\dfrac{d}{dx}\,.\,x\,.\,y$ we proceed from right to left; $x\times$ or $x\,.$ is now clearly an order "multiply by x". It would seem reasonable then to keep the same rule in Group Theory. *E.g.* if Rx is $\dfrac{1}{x}$ and Sx is $1-x$, then RSx is $\dfrac{1}{1-x}$ and SRx is $1-\dfrac{1}{x}$. Boys will probably first meet this in connexion with cross-ratio. But here confusion again enters, for some books treat RS as "R, followed by S" and not as "S operated on with R". This is now becoming a very March hare! I give it up!

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H.M.C.

FUNCTIONAL ANALYSIS

BY J. L. B. COOPER

Functional analysis is the study of functions defined on certain abstract sets; these sets may vary greatly in their structure and the nature of their elements but they have in common the property that it is possible to define in them the notions of convergence, limit and so on: in other words, they are topological spaces.* Most of the spaces studied in functional analysis have in addition algebraic properties: that is to say, addition and sometimes multiplication of their elements is possible.

From one point of view, functional analysis is a generalisation of classical analysis: it differs from it in so far as classical analysis studies only functions defined on the real line, the complex plane, or n-dimensional spaces; and from this point of view its study is part of a general tendency to abstraction and axiomatisation which is a feature of modern mathematics. To say this alone, however, is to miss an important point: that the study of functional analysis has been one of the incentives to, rather than a product of, the drive to generalisation in mathematics: indeed, analysts, as a class, are of all mathematicians the least affected by an interest in axiomatics; what first gave functional analysis its importance was its role as a tool in the study and development of classical analysis rather than its position as an abstract version of it.

This can best be illustrated by some examples of the problems which led to the study of the subject. Early examples are those in the calculus of variations. In this, one is given an integral involving an unknown function, and one tries to find a function which makes the integral a minimum. The classical procedure here is to follow the lines of the theory of maxima and minima of ordinary functions; one calculates the change in the integral for a small variation of the function, and finds when the variation is zero to the first order; this is the way in which Euler's differential equations are derived. Now, if one writes the integral

$$I(\phi) = \int f(\phi, \phi', \ldots, x, y, z)\, dx\, dy\, dz$$

then I is a function defined on a set whose members are the functions ϕ.

A function, such as I, whose values are ordinary numbers, is called a functional: and this is the origin of the name " functional analysis ". In the calculus of variations one is, in effect, finding the gradient of I with respect to ϕ: the gradient of I in the direction of an increment ψ is

$$D_\psi I(\phi) = \lim_{\epsilon \to 0} \frac{I(\phi + \epsilon\psi) - I(\phi)}{\epsilon}$$

If, for example, I is the integral of Dirichlet in potential theory

$$I(\phi) = \int [\nabla . \phi]^2\, dx\, dy\, dz \quad \ldots\ldots\ldots\ldots (1)$$

then the gradient of ϕ in the direction of an increment ψ is

$$D_\psi I(\phi) = 2\int \nabla^2 \phi \,.\, \psi\, dx\, dy\, dz$$

and so the condition that I should be a minimum, subject to suitable conditions on the value of ϕ on the boundary of the region of integration is

$$\nabla^2 \phi = 0 \quad \ldots\ldots\ldots\ldots (2)$$

Now this calculation led Riemann to an error which was almost as fruitful as his error concerning the zeroes of the Zeta function: he argued that the

* An apparent exception to this is measure theory, in which functions are defined on sets with a measure, not a topology.

integral (1) must have a minimum, and (2) must have a solution, with any given values of ϕ on the boundary of the region D.

This reasoning is due to a transference to functional analysis of the familiar fact that a function on a bounded set must attain a minimum on the set. The transference could not be justified by any means known to mathematicians at Riemann's time. What was needed for its justification was a theory of limits and convergence on spaces made up of functions, and analogous abstract spaces: a theory whose necessity was explicitly foretold by Riemann, and which began to be constructed at the beginning of this century. It was not until 1900 that a valid version of Riemann's argument was given by Hilbert.

A second precursor of functional analysis can be found in the studies of Operational Calculus; the well known wangles with D by which we solve ordinary and partial differential equations. The idea that this subject is the study of a function—the operator D—whose range of definition is a set of functions is to be found in the work of British mathematicians of the1840's, notably of Murphy: it is the first formulation of one of the essential ideas of functional analysis, and the origin of the modern theory of linear operators.

These examples illustrate a feature common to many branches of functional analysis. In the beginning functional analysis played a heuristic role, suggesting results by presenting analogies between the relations being considered and those of ordinary geometrical space (in the case of calculus of variations, for example) or of ordinary algebra (in the case of operational calculus) but could not provide accurate proofs. For these, a theory of limiting processes in abstract spaces was needed, a theory which was developed from the beginning of this century onward, and goes under the name of general topology.

For a long time functional analysis made use almost exclusively of one branch of this theory—the theory of metric spaces: A metric space is a set S consisting of objects, say $x, y, z, \ldots$ for which a distance is defined: that is to say, to any two objects x, y is associated a number $d(x, y)$ which has the properties that

(i) $d(x, y)$ is never negative, and is zero only if $x=y$;

(ii) $d(x, y)=d(y, x)$;

(iii) $d(x, z)\leqslant d(x, y)+d(y, z)$ (The triangle inequality).

Convergence in a metric space is defined by saying that

$$x_n \to x \text{ as } n\to\infty \text{ if } d(x_n, x)\to 0 \text{ as } n\to\infty.$$

We can now define other topological notions to correspond: a set is open if, when x is in the set, all points within some positive distance of x are also in it: a set is closed if it contains all its limit points, etc.

The real and complex numbers, n dimensional Euclidean space, any surface, curve or any other subset of these spaces: all these are metric spaces with the ordinary definition of distance. Other examples of metric spaces are:

(1) S. The space of all numerical sequences, where when x is the sequence (x_n), and y the sequence (y_n),

$$d(x, y)=\sum\frac{|x_n-y_n|}{2^n[1+|x_n-y_n|]}.$$

(2) B. The space of all bounded sequences, where

$$d(x, y)=\sup|x_n-y_n|.$$

(3) C. The space of all continuous functions defined on $(0, 1)$, with

$$d(x, y)=\max|x(t)-y(t)|,$$

where x and y stand for the functions $x(t)$ and $y(t)$.

The notion of convergence in this last space is the familiar notion of uniform convergence : $d(x_n, x)\to 0$ means precisely that

$$x_n(t)\to x(t) \text{ uniformly in } t \text{ as } n\to\infty.$$

If a sequence (x_n) of points of a metric space has the property that $d(x_n, x_m)\to 0$ as $m, n\to\infty$, then the sequence is called a Cauchy sequence. Any convergent sequence in a metric space is a Cauchy sequence. In a Euclidean space of any finite number of dimensions any Cauchy sequence converges to a point of the space. On the other hand, in the space of rational numbers, nonconvergent Cauchy sequences exist : for example, the sequence of sums of the decimal expansion of π. A space in which every Cauchy sequence is convergent is called a complete space. It is quite easy to see that the spaces S, B, and C are complete.

These ideas suffice for many important results in functional analysis. I shall give one example ; the proof of an important fixed point theorem, which is due to the great Polish mathematician Banach.

The theorem is as follows. Let E be any complete metric space. Let A be a function defined on the space, taking its values in the space, and having the property that there is a number θ such that $0<\theta<1$ and that, for any two points x, y of the space,

$$d(A(x), A(y))\leqslant \theta\, d(x, y).$$

Then there is just one fixed point x for A : i.e. just one point such that $A(x)=x$.

The proof is simple. Starting with an arbitrary point x_0, we define successively $x_1, x_2, \ldots$ by $x_{n+1}=A(x_n)$. Then

$$d(x_{n+1}, x_n)\leqslant \theta\, d(x_n, x_{n-1})\leqslant \ldots\ \theta^n\, d(x_1, x_0),$$

and so

$$d(x_n, x_{n+p})\leqslant d(x_n, x_{n+1})+\ldots\ d(x_{n+p-1}, x_{n+p})$$
$$\leqslant(\theta^n+\theta^{n+1}+\ldots)\, d(x_0, x_1)<\frac{\theta^n}{1-\theta}\, d(x_0, x_1)$$

whence it follows that (x_n) is a Cauchy sequence, and so converges to a point x. Clearly $Ax=x$, and if $Az=z$ then $d(x, z)\leqslant\theta\, d(x, z)$ so that $d(x, z)=0$ and $x=z$.

Among the applications of this result is Newton's method of solving equations. Of more interest is its application to the proof of the existence of a solution of the differential equation

$$\frac{dy}{dx}=f(x, y) \quad \text{with } y(x_0)=y_0$$

where f is continuous and $|f(x, y_1)-f(x, y_2)|<K|y_1-y_2|$ for some constant K. Here we write $Ay=z$ where

$$z(x)=y_0+\int_{x_0}^{x} f[\xi, y(\xi)]\, d\xi,$$

so defining a function on the space of continuous functions on some interval (x_0, x_0+h), which, provided h is not too large, satisfies the requirements of the theorem ; Picard's proof of this existence theorem is thus a case of the general result.

In this theorem we are not concerned with the algebraic properties of the space : but the major part of modern work is concerned with spaces with an algebraic structure, and particularly with linear spaces : those in which any two elements can be added and can be multiplied by a number to give another element of the space. A function A is called a linear function, or, more usually,

a linear operator, if it is defined and has its values in a linear space and if

$$A(\lambda x + \mu y) = \lambda A(x) + \mu A(y).$$

when x, y are any elements of the space, λ, μ any numbers.

The operator of differentiation is an example of a linear operator.

Hilbert Space. The most important of the spaces studied in functional analysis is Hilbert space. This is a direct generalisation of Euclidean space. It may be defined axiomatically as a linear space H in which multiplication by complex numbers is defined and in which a hermitian scalar product is defined, that is to say, for every pair of members x, y of H there exists a number, written (x, y) with the properties

$$(\lambda x_1 + \mu x_2, y) = \lambda(x_1, y) + \mu(x_2, y),$$
$$(x, y) = \overline{(y, x)}$$
$$(x, x) > 0 \quad \text{if} \quad x \neq 0.$$

The hermitian scalar product resembles the scalar product of vector analysis: if instead of a complex Hilbert space we consider a real one, in which only multiplication by real numbers is allowed, the scalar product takes real values only and the resemblance is even closer. The magnitude or norm of a vector in the space is defined by

$$\| x \| = \sqrt{(x, x)}$$

and the distance between two vectors is taken to be $\| x - y \|$. With this definition of distance, Hilbert space is required to be complete.

Two examples of Hilbert spaces are

(1) The space l^2, which consists of all sequences $x = (x_n)$ such that

$$\sum_1^\infty | x_n |^2 < \infty, \text{ and in which we take}$$

$$(x, y) = \sum_1^\infty x_n \bar{y}_n \quad \text{when } x = (x_n),\ y = (y_n)$$

(2) The space $L^2(a, b)$, where (a, b) is any interval of the line, and an element x of L^2 is a class of functions such that any two members of the class are equal almost everywhere, and if $x(t)$ is a member of x then

$$\| x \|^2 = \int_a^b | x(t) |^2 \, dt \text{ is finite.}$$

We take
$$(x, y) = \int_a^b x(t) \overline{y(t)} \, dt.$$

Hilbert space was introduced by Hilbert in his study of integral equations. An integral operator K, defined by $y = Kx$ where

$$y(u) = \int_a^b K(u, t)\, x(t)\, dt, \quad K(u, t) = \overline{K(t, u)},$$

transforms Hilbert space into itself and behaves in many ways like a symmetric linear transformation in an ordinary n-dimensional Euclidean space if $K(u, t)$ satisfies certain conditions : for instance if

$$\iint | K(u, t) |^2 \, du\, dt < \infty \quad \text{..............................(3)}$$

a condition which is satisfied whenever $K(u, t)$ is a continuous function and (a, b) a finite interval. In these circumstances the operator K can be put in a form which corresponds to the diagonal form for a matrix ; it turns out that there is a set of elements $(\phi_n(x))$ in L^2 which are orthogonal and of unit norm :

that is to say

$$\int \phi_m(t)\overline{\phi_n(t)}\,dt = 1 \quad \text{if } m = n,$$
$$= 0 \quad \text{if } m \neq n,$$

such that, for real numbers λ_n

$$K\phi_n = \lambda_n \phi_n$$

and
$$K(u, t) = \Sigma \lambda_n \phi_n(n)\overline{\phi_n(t)}$$

(the series being convergent in the sense of distance in L^2).

A similar theory holds for some differential operators: for instance, consider the operator $T = d^2/dx^2$ operating on the functions of $L^2(0, 2\pi)$ subject to the boundary conditions $f(0) = f(2\pi)$, $f'(0) = f'(2\pi)$. Then for any integer n, the functions

$$\phi_n(x) = e^{inx}/\sqrt{2\pi}$$

satisfy
$$T\phi_n = -n^2\phi_n$$

and satisfy the boundary conditions: they are called the eigenfunctions of the operator. Any function of $L^2(0, 2\pi)$ can be represented as a series of these eigenfunctions, the wellknown Fourier series:

$$f(x) = \Sigma f_n \phi_n(x) = \frac{1}{\sqrt{2\pi}} \Sigma f_n e^{inx} \qquad \text{.........................(4)}$$

where
$$f_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} f(x) e^{-inx}\,dx$$
$$= \int f(x)\overline{\phi_n(x)}\,dx,$$

and where

$$\| f \|^2 = \int_0^{2\pi} | f(x) |^2\,dx = \Sigma \,| f_n |^2 = \| (f_n) \|^2.$$

One interesting consequence of this result is that a space L^2 and the space l^2 are isomorphic; just as any two 3-dimensional Euclidean spaces are isomorphic. Every element of L^2 has corresponding to it a sequence from l^2, its Fourier coefficients, and vice versa.

For some other differential operators, or for integral operators which do not obey the condition (3), the analogies with operators in finite dimensional spaces are not so close. The demands of the theory of differential operators, and more specifically the requirements of quantum theory, have led to a very complete theory of one class of operators, the selfadjoint operators; this theory is due to von Neumann; it differs from the theory just explained in as much as one needs integrals of, as well as sums of eigenfunctions.

For example, consider the operator d^2/dx^2 defined over the entire real line. The solutions of

$$\frac{d^2}{dx^2}\phi = \lambda\phi$$

are the functions $e^{i\sqrt{\lambda}x}$: these are not members of $L^2(-\infty, \infty)$ and so are not, properly speaking, eigenfunctions, but the result corresponding to the expansion (4) is the expansion of a function as a Fourier integral

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{iux}\,du. \qquad \text{..............................(5)}$$

In terms of this theory, one can construct an operational calculus, that is to say, one can define precisely what is meant by functions of an operator. For

example, a function ϕ of the operator d^2/dx^2 is the operator which takes $f(x)$, in equation (5) into the function

$$\phi\left(\frac{d^2}{dx^2}\right) f(x) = \int_{-\infty}^{\infty} \phi(-u^2)F(u)e^{iux}\, du$$

whenever this integral is defined.

Banach Spaces. Hilbert space is a particular case of a wider class of linear spaces, which are called Banach spaces. These are complete metric linear spaces, in which the metric is defined by a norm $\| x \|$ which is positive, zero only for $x=0$, and satisfies

$$\| \lambda x \| = | \lambda | \| x \|$$

for every x and every complex number λ,

$$\| x+y \| \leqslant \| x \| + \| y \| .$$

Examples of Banach spaces are the spaces l^p, L^p for $p \geqslant 1$ where l^p consists of all sequences $x=(x_n)$ with

$$\| x \|^p = \Sigma \mid x_n \mid^p$$

and L^p of functions $x(t)$:

$$\| x \|^p = \int \mid x(t) \mid^p dt.$$

The space of continuous functions with $x = \sup \mid x(t) \mid$ is another Banach space.

Banach algebras. There is a most interesting theory of those Banach spaces in which not only addition but also multiplication of the elements is possible ; these are called Banach algebras. The space of continuous functions is an example : the product of two continuous functions is a continuous function and since

$$\sup \mid x(t)\, y(t) \mid \leqslant \sup \mid x(t) \mid \sup \mid y(t) \mid$$

it satisfies the condition

$$\| x\, y \| \leqslant \| x \| \, \| y \|$$

which is usually required to hold in a Banach algebra. The theory of commutative Banach algebras is due to the Russian mathematician, Gelfand. One of his main results is that, in effect, this example of the algebra of continuous functions is typical of what happens in general : if R is any commutative Banach algebra which has a unit, that is, an element e such that $ex=x$ for every x in the algebra, then there is a space T (compact and Hausdorff) and a one-one correspondence of the elements of R with continuous functions on T ; to each x in R corresponds a continuous function, say $f_x(t)$, on T, xy corresponds to $f_x(t)\, f_y(t)$.

Gelfand made a most elegant use of this to provide a proof of a theorem of Wiener, which is the basis of Wiener's Tauberian theorem. The set of functions which are sums of absolutely convergent Fourier series

$$x(t) = \Sigma x_n e^{int}$$

with

$$\| x \| = \Sigma \mid x_n \mid$$

is a commutative Banach algebra when multiplication is defined in the ordinary way. The continuous function corresponding to the element x is $x(t)$ itself : and it follows from Gelfand's theory that $x(t)$ has an inverse in the algebra if $1/x(t)$ is continuous : so that we get Wiener's theorem that the inverse of an absolutely convergent Fourier series which does not vanish anywhere is absolutely convergent.

This theory is closely related to the spectral theory of operators. If H is a bounded selfadjoint operator in Hilbert space, the set of functions of H forms

a Banach algebra. To this corresponds, according to Gelfand's theory, a set of continuous functions on a space T; and this space T turns out to be the spectrum of H, a function $f(H)$ therefore corresponds to a numerically valued function on the spectrum of H.

Another, less obvious example, is the space $L(-\infty, \infty)$, the space of functions summable over $(-\infty, \infty)$, in which a type of multiplication, called convolution, is defined as follows: if f and g are two summable functions, their convolution is the function $h=f * g$, where

$$h(x)=\int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt.$$

It is easy to see that $f * g = g * f$, and that

$$\| h \| = \int_{-\infty}^{\infty} | h(x) |\, dx = \int dx \int | f(x-t)\, g(t)\, dt |$$

$$\leqslant \int | g(t) |\, dt \int | f(x-t) |\, dx = \| f \| \, \| g \|,$$

so that this is indeed a Banach algebra, commutative, but with no unit element. The theory of this Banach algebra provides another approach to the Tauberian theorems, and also to the theory of the Fourier transform: for the complex valued function associated with a function f in this algebra is its Fourier transform, $F(u)$: in fact, the Fourier transform of the convolution of two functions is the product of the Fourier transforms of the original functions.

This last argument, again, can be carried over from the theory of functions of a real variable to the theory of functions on topological groups, and it enables a theory of Fourier transforms and harmonic analysis on topological groups to be established, which includes not only the ordinary theories of Fourier series and integrals but also a variety of theories concerned with the expansion of an arbitrary function in terms of functions which are invariants under some group: for example, the spherical harmonics are invariant in the rotation group and the theory of harmonic analysis on the rotation group leads to the theory of expansions in spherical harmonics.

The theory of distribution. An important recent development of functional analysis is the theory of distributions which we owe to Laurent Schwartz. Distributions are constructed in order to generalise the concept of function. The motives for the construction of distributions are these: in many branches of mathematics we need to introduce what are sometimes called improper functions. For example, the displacement of a vibrating string satisfies a differential equation which enables us to calculate its condition at any time if we are given its initial condition: if in its initial condition the string is V-shaped, we can still give its later shapes—which are also V-shaped—but these do not satisfy the equation since the functions involved are not differentiable at the point of the V. The Heaviside unit function, $H(x)$, which is 1 if x is positive and 0 if x is negative, has no derivative at 0; yet its derivative, which is called the Dirac delta function $\delta(x)$, is used extensively in quantum theory: it has the property that for every continuous function f

$$\int f(x)\, \delta(x)\, dx = f(0), \qquad \text{.................................(6)}$$

a property possessed by no genuine function.

In order to legitimize these usages, Schwartz has devised the following construction, which I shall explain for the case of functions of one real variable. Let D denote the set of all functions of a real variable which vanish outside a bounded set and are differentiable infinitely often. If f is any continuous function, then there is a linear functional T_f on D corresponding to f; defined by

$$T_f(\phi)=\int_{-\infty}^{\infty} f(x)\, \phi(x)\, dx.$$

If f is differentiable, then on integration by parts,

$$T_{f'}(\phi) = \int f'(x)\phi(x)\,dx = -\int f(x)\phi'(x)\,dx = -T_f(\phi').$$

Thus the operation of differentiating f corresponds to the operation changing $T_f(\phi)$ into $-T_f(\phi')$. Now $T_f(\phi')$ is always defined and is also a linear functional on D; so that even if f has no derivative, this functional can stand for its derivative. Functionals of this type represent the improper functions: for instance the functional representing the Heaviside unit function is

$$T(\phi) = \int_0^\infty \phi(x)\,dx$$

and so that representing the delta function is defined by

$$-T(\phi') = -\int_0^\infty \phi'(x)\,dx = \phi(0).$$

This is the legitimate version of the improper identity (6).

The ability to use the differential operator freely gives this method great power in the theory of differential equations, and forms one of the best methods of setting the theory of the operational calculus for differential operators on a sound basis.

These examples will I hope, show how functional analysis can assist analysis by the guidance of geometrical intuition, and can throw new light on old theories by presenting them in a broader framework. There are many interesting theories which space and the need for more recondite techniques makes it impossible to discuss here. The general theory of topological vector spaces, with its interesting applications of the simple geometrical notions of convexity is outstanding among these. For the reader interested in these matters, I suggest some books below.

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University College, Cardiff J. L. B. C.

GLEANINGS FAR AND NEAR

1925. James Killian expressed it (the increasing separateness of science in the modern world) more bluntly by reporting two intelligences that have been making the rounds of the faculty lounges—one the observation that the scientist knows nothing of the liberal arts and regrets it, while the humanist knows nothing of science and is proud of it. The other was an incident said to have occurred at a liberal arts faculty meeting. When a student named Cicero was reported to have flunked Latin, everybody laughed, but when a student named Gauss was reported to have failed mathematics, only the science professors laughed.—William S. Beck: *Modern Science and the Nature of Life.* [Per Dr. H. Martyn Cundy.]

CLASS ROOM NOTES

27. "Peg-board" as a Visual Aid.

I wonder how many of my colleagues have already investigated the possibilities of the material known as peg-board as a teaching aid? Probably everyone knows this perforated hardboard, now in such common use, and there is no need to describe it here. With the addition of a few pieces of easily acquired equipment a sheet of it can be a very useful adjunct to a mathematical classroom.

In the first place it can largely supersede the usual squared blackboard for instruction in the elements of graph work, so let us look at this first.

In addition to a conveniently sized piece of board we need two appropriately long laths, drilled with holes at intervals to correspond with those on the board, a number, the more the better, of pegs, and possibly a length of elastic thread purchasable at any haberdashery store. The pegs are better for a shank and larger head as the holes in the board are but small; if they can have their heads dyed in various colours all the better.

The functions of these extras are obvious, I think; the laths can be pegged in any desired positions to form axes, the pegs are the "points" and the thread will join them. (As a rule the imagination of the pupil will complete the curve without this thread, but there are occasions when it will be useful.)

Now that Grammar School classes appear to be reaching the three dozen size, the business of getting round to see that each pupil can mark a point correctly on his squared paper is a lengthy one; but it must be done, for it is not as easy a job for all of them as might appear, and I have always found plenty who can make errors. With the board in front of them, the axes in position and units chalked on the board, and a supply of pegs, the abilities of a whole class can be tested in a short time. Each pupil sees the efforts of his fellows and learns from them. Position-fixing and the idea of co-ordinates can thus be easily grasped, and, this done, the step to graph "construction" is short and easily taken. Building a graph in this way by sticking in pegs instead of making "dots" on paper makes it a much more real business. Further it is a co-operative job; each pupil is keenly interested in the success of the rest and the blunderer is soon put right.

Before proceeding to algebraic linear graphs a great deal of work can be done with "travel graphs"—cycle journeys plotted and interpreted, simple railway time-tables constructed, the "idiot in the bath-room" activities exposed etc.; all such can be very rapidly put on the board. It is also an advantage that an incorrectly placed peg is removed and replaced without the use of a duster and the resultant smear.

Linear algebraic functions can be investigated with ease and rapidity. The meaning of the intercept constant is much more clear when a real peg is moved up or down in full view of all; the slope or gradient similarly can be made more obvious.

The transfer of work from board to individual squared paper should present little difficulty if enough collective practice has been given.

The converse problem of "given the graph, find the function" clearly gives us much scope; it will not be long before the row of pegs stretched across the board truly represents a line whose equation can be quickly written down by most, if not all.

Parabolic graphs will probably need some preliminary preparation by the teacher since non-integral values will often occur. But a parabola suggested by a string of coloured pegs conveys a much more vivid image than dots on paper can ever do. Further, the nature of the turning point can be shown if the elastic thread is used; flat-topped and sharp-pointed curves will be things of the past very soon, for the eye completes the curve naturally.

In general the board will be of great service in introducing a new type of function, for exploratory work and for getting an idea of the graph before committing it to paper, since errors can so easily be corrected without affecting the correct parts of the work.

Younger pupils will enjoy putting in the pegs, the middle school will appreciate having a pre-view of the graph, and I imagine that many a sixth-former will teach himself something of value about the effects of a change of axes or origin, for instance, with the aid of the board and a few pegs.

The foregoing is most certainly not exhaustive, but it may be suggestive. At least I hope so.

In elementary Geometry, too, the board and pegs can be helpful; simple loci are an obvious use for them—two pegs and a large set-square will mark out the " segment locus ", and a piece of cord between two foci, the ellipse. With additional laths the properties of transversals across parallels can be demonstrated with the advantage of being able to rotate the transversal into new positions while the class *sees* the changes in the corresponding and alternate angles; the arrangement will be similarly effective in establishing the equality of these angles as a test for parallelism, since it is so easy to change to non-parallel lines. The availability of equally spaced intervals along two lines gives some scope for the construction of some simple envelopes.

With the board placed horizontally and the use of thin dowel rods or straight canes, some of the difficulties of 3-D work in Mensuration and Trigonometry can be removed.

Once again, this note is no more than suggestive; I have by no means yet fully explored the possibilities, and I hope to hear of further experiments with this material by some of those who will be stimulated to give it a trial.

Chingford County High School P. J. SMITH

28. A proof by the Ratio-formula with oblique axes of the theorem:

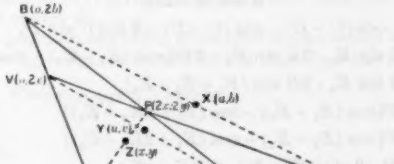

FIG. 1.

" The mid-points of the three diagonals of a quadrilateral are collinear."

Taking as axes of coordinates the sides OA, OB of the quadrilateral, whose remaining vertices are P, U and V, let these vertices have the coordinates shown. Then the mid-points X, Y, Z are, by the formula, (a, b), (u, v), (x, y) respectively.

If P divides AV in the ratio $\lambda : 1$ and

BU ,, ,, ,, $\mu : 1$

then by the formula $2x = \dfrac{2a}{\lambda + 1} = \dfrac{2\mu u}{\mu + 1}$, each of which $= \dfrac{2(a - \lambda\mu u)}{1 - \lambda\mu}$

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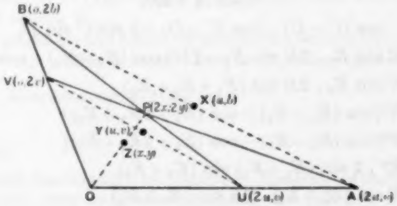

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then by the formula $2x = \dfrac{2a}{\lambda + 1} = \dfrac{2\mu u}{\mu + 1}$, each of which $= \dfrac{2(a - \lambda\mu u)}{1 - \lambda\mu}$

$$\text{and}\quad 2y=\frac{2\lambda v}{\lambda+1}=\frac{2b}{\mu+1},\ \text{each of which}=\frac{2(b-\lambda\mu v)}{1-\lambda\mu}$$

$$\therefore\ x=\frac{a-\lambda\mu u}{1-\lambda\mu},\quad y=\frac{b-\lambda\mu v}{1-\lambda\mu}$$

so that $Z(x, y)$ is the point dividing XY in the ratio $-\lambda\mu : 1$ i.e. in the ratio $-\frac{ab}{uv} : 1$.

Hence Z lies in XY.

Chichester High School for Boys O. R. HULBERT

29. Ptolemy's Theorem.

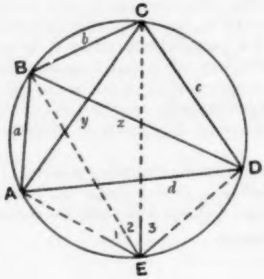


FIG. 1.

Formulae used :

$$a/\sin A=2R,$$

$$\cos(C-D)-\cos(C+D)=2\sin C\sin D.$$

$$\begin{aligned}
ac&=2R\sin E_1\,.\,2R\sin E_3=2R^2[\cos(E_1-E_3)-\cos(E_1+E_3)]\\
bd&=2R\sin E_2\,.\,2R\sin(E_1+E_2+E_3)\\
&=2R^2[\cos(E_1+E_3)-\cos(E_1+2E_2+E_3)]\\
\therefore\ ac+bd&=2R^2[\cos(E_1-E_3)-\cos(E_1+2E_2+E_3)]\\
&=2R^2\,.\,2\sin(E_1+E_2)\sin(E_2+E_3)\\
&=2R\sin(E_1+E_2)\,.\,2R\sin(E_2+E_3)\\
&=xy.
\end{aligned}$$

Hawarden Grammar School JAMES BELL

30. Parabolic sections of a circular cone.

In a large proportion of books which deal with the derivation of conics as sections of a circular cone, the following statement is made (with slight difference in wording) :

(A) " If the cutting plane is parallel to one of the generators of the cone, the section is a parabola ".

Examples of books in which the statement appears are : Taylor, *Geometry of Conics* (1891, p. 79) ; Smith, *Geometrical Conics* (1894, p. 182) ; Courant and

Robbins, *What is Mathematics?* (1943, p. 199); Richardson, *Fundamentals of Mathematics* (1949, p. 244); Kline, *Mathematics in Western Culture* (1954, p. 47); Ogilvy, *Through the Mathescope* (1956, p. 66).

Statement A is untrue. If the cutting plane is parallel to one of the generators of the cone, the section is in general a hyperbola, and the cutting plane is also parallel to a second generator. Statement A could be made true by saying " one and only one " instead of " one ", or by saying that the plane is also perpendicular to the plane containing the axis and the generator concerned. However, the simplest way is probably to say that the section is a parabola if the plane is parallel to a tangent plane : this is done by Whitehead, *Introduction to Mathematics* (1911, p. 130) and Sommerville, *Analytical Conics* (1933, p. 36A).

Durell, *A Concise Geometrical Conics* (1927, p. 78) says " The plane of the section in this case [where the plane makes an angle α with the axis, α being the semi-vertical angle] is parallel to a generator of the cone and we obtain a parabola!" This is true, but the statement that the plane is parallel to a generator is a little misleading.

Allendoerfer and Oakley, *Principles of Mathematics* (1955, p. 263) say " The parabola can be obtained by cutting a right circular cone with a plane parallel to a generator." This is also strictly true, but very misleading, because it suggests that only a parabola can be obtained in this way (there is no statement like Durell's in square brackets above).

Finally, Kasner and Newman, *Mathematics and the Imagination* (1949, p. 107) say " The parabola is formed by the section of a cone cut by a plane parallel to the opposite edge." What does " opposite edge " mean?

It should in fairness be said that in the books listed it is usually clear from a diagram what is intended.

E. J. F. PRIMROSE

31. Quadric: plane of the section having a given centre.

When $s=0$ is the equation of a quadric surface, the work of Class Room Note 24 (*Gazette* 341, p. 211) gives the plane which cuts the quadric in a conic having prescribed centre P_1.

For if P_2 is any point on the conic of section whose centre is P_1, then we find that P_2 lies on $s_1=s_{11}$. This is linear, and is therefore the required plane.

Technical College, Kingston-upon-Thames F. GERRISH

32. The Slide Rule.

The Incorporated Association of Assistant Masters' report on *The Teaching of Mathematics* is not very enthusiastic about the value of teaching the use of the slide rule in schools. It suggests that " boys should be shown how to construct a simple slide rule and should learn how to use it, at any rate, for simple multiplication and division ", but does not recommend its general use in the classroom although " there is much to be said for the use of the slide rule as a practical aid to laboratory calculations." This view tends to overlook the fact that the main value of the slide rule lies not so much in the arithmetical fields but in its use in solving trigonometrical problems. There is much work in Mechanics, especially when dealing with vector quantities, when the application of the appropriate principle of Mechanics reduces the problem to the solution of one or more triangles. The numerical work, often long and tedious, adds nothing to the understanding of the mechanical principles and can easily be avoided by the use of a slide rule. The question of degree of accuracy also referred to in the report does not arise to any serious extent, as the solution of

this type of problem is often recommended by graphical methods and the slide rule should normally produce a higher degree of accuracy than scale drawing.

The majority of textbooks on the slide rule start with a brief description of the scales and go straight on to the fundamental operations of multiplication and division, followed in most cases by examples involving continuous operations. Many students, especially those with limited mathematical ability, make little progress to begin with as errors in reading the scales lead to incorrect answers which in turn weaken their confidence in the slide rule. In addition these students seldom appreciate that many of the operations are general and can be used with different combinations of scales.

As many different types of slide rule are in use, it is assumed in what follows that the slide rule possesses the minimum essentials of A, B, C, and D scales together with the trigonometrical scales S and T.

In designing a teaching sequence to introduce the slide rule at the start of a sixth form course in mathematics or engineering attention should be paid to the following points :

(i) The slide rule is a computing instrument which provides practically the same information as the usual set of mathematical tables, but not to the same degree of accuracy.

(ii) Adequate practice in reading the scales from direct settings is essential before any operations are introduced.

(iii) The generality of many operations should be continually emphasised—e.g. the same operations but on different scales are used to calculate 7×17 and $7 \times \tan 17^\circ$.

(iv) The principle of proportionality of corresponding settings is of fundamental importance.

(v) Probably the greatest value of the slide rule is the ease with which the solution of triangles can be carried out, and consequently this should be introduced as early as possible.

With these points in mind it is suggested that the following order of treatment has some advantages over the more conventional method.

(1) *Introduction*. The slide rule provides all the information that the normal set of tables provides e.g.

(*a*) Squares and square roots.

(*b*) Sines and cosines.

(*c*) Tangents (up to 45° to begin with).

All these can be read off by direct settings which are illustrated below. Care must be taken to put the decimal point in the correct place.

The student can practise these settings as much as he likes, checking his answers from tables. In doing so he will soon appreciate the degree of accuracy which can be obtained from a slide rule.

(2) *The first operation*. The first operation introduced is the simplest, that of finding reciprocals. This is a general principle and the settings are illustrated for ordinary numbers and for tangents. It will be noted that the ends of the scales are interchangeable.

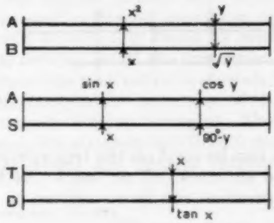
Combinations of the A and B scales, or A and S scales can equally well be used, and the student should practise finding secants, cosecants, and cotangents.

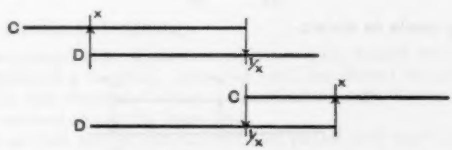
Using the relationship $\tan x = 1/\tan (90^\circ - x)$ the tangents of angles greater than 45° can be dealt with by the following setting :

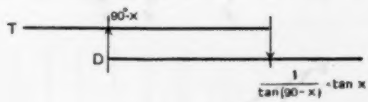
It is advisable at this point to deal with the problem of tangents of small angles, and tables can be used to illustrate that $\tan x = \sin x$ to within slide rule accuracy for angles up to about 6°.

(3) *Multiplication*. The most elementary of all the mathematical tables are the multiplication tables, and the slide rule can be converted into a (say) four times table by the settings :

Once again it should be noted that either index on the slider can be used; that the setting is universal, and can be used with the following combinations of adjacent scales—*A* and *B*, *A* and *S*, and *T* and *D* (illustrated below).



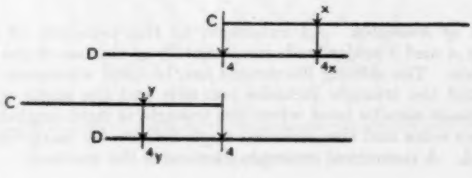


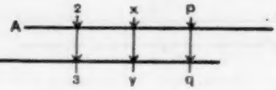


**(4) *Conversion Tables.* The multiplication table idea leads straight to the general principle of proportional settings on corresponding scales.
In the above setting the relationship**

$$\frac{2}{3}=\frac{x}{y}=\frac{p}{q}$$

holds and similar settings can be used on the *A* and *B* scales, or *C* and *D* scales to make conversion tables for money, percentages, miles per hour to feet per second, and many other examples.

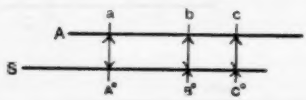


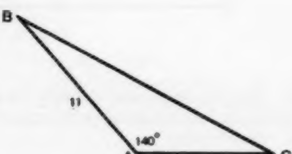


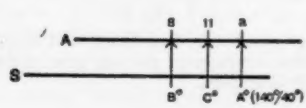
The principle of proportion can be used on the trigonometrical scales for the solution of equations such as $\sin x = 45/67$ or $\tan x = 34/87$. Using the fact that $\tan 45° = 1$ the latter expression can be re-written as

$$\frac{\tan x}{34} = \frac{\tan 45°}{87}$$

and the setting made as shown :







(5) *Solution of triangles.* An extension of the principle of proportional settings on the *A* and *S* scales leads immediately to the use of the sine formula on the slide rule. The setting illustrated can be used whenever the information given about the triangle includes one side and the angle opposite to it. The setting should also be used when the triangle is right-angled.
The case of two sides and the included angle follows by using the " trial and error " method. A numerical example illustrates the method.

In the triangle ABC, $B+C=40°$, so the slider is adjusted by trial and error until the readings on the S scale opposite the values 8 and 11 on the A scale add up to 40°. The values are approximately $B=16° 40'$ and $C=23° 20'$. The length of the side BC is then read opposite 40°, as sin 40° = sin 140°.

(6) *Division.* It will appear strange that so much has been attempted and no reference made to such a simple operation as division. There are several points in this sequence at which it could be introduced—e.g. as an extension of the multiplication setting, or of the principle of proportionality.

It is considered that this method of approach to the slide rule will give the normal student confidence in the accuracy of the instrument and in the ease with which it can be used to solve triangles. Once he is convinced that it is a reliable and useful aid, the student will find it much easier to master the more complicated settings which are usually introduced much too early in the course.

Welbeck College, Worksop, Notts. G. R. LANGDALE

33. To draw a tractrix and catenary.

It is often supposed that these curves cannot be drawn without the aid of tables. The following method, however, will be found to give remarkably accurate results, the accuracy being in fact limited only by the distance apart of the points marked along the base-line.

Draw the base-line across the foot of the paper and mark points on it at equal intervals of, say, 0·2 in., beginning near the left-hand margin and continuing as far as the middle of the paper. Number these points 0, 1, 2, 3 ..., from left to right. Through points 1, 3, 5 ... draw lines at right angles to the base-line. (These will be called " vertical lines 1, 3, 5... ".) With points 0, 2, 4 ... as centres draw quadrants of circles of fixed radius c (say 2 in.), as shown in the Figure. (These will be called " quadrants 0, 2, 4 ... ".)

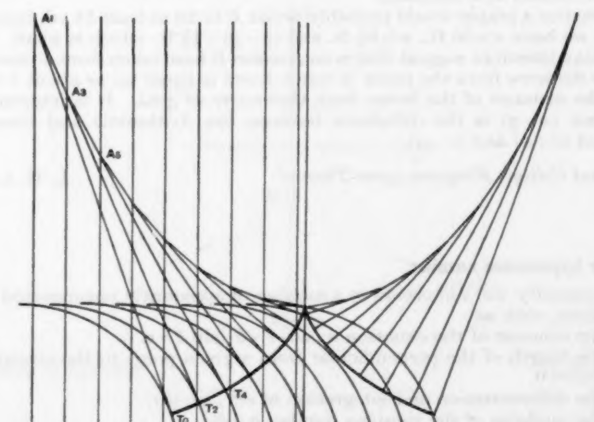


FIG. 1.

On the vertical line 1, choose a point A_1, as far from the base-line as the compasses will conveniently span, and from it draw a tangent A_1T_0 to quadrant 0. (The point of contact T_0 must be accurately determined.) With A_1 as centre draw an arc of a circle from T_0 to meet quadrant 2 at T_2. Join A_1T_2, cutting vertical line 3 at A_3. With centre A_3 draw an arc from T_2 to meet quadrant 4 at T_4. Join A_3T_4, cutting vertical line 5 at A_5; and so on.

The points $T_0, T_2, T_4 \ldots$ will lie on a curve approximating to a tractrix, and the points $A_1, A_3, A_5 \ldots$ will lie approximately on a catenary. When the tractrix has been continued until it meets the catenary (at a point distant c from the base-line) the vertex of both curves has been reached. A line drawn through this point at right angles to the base-line is an axis of symmetry for both curves and should be used as such for drawing the other half of each of them.

The Old School, Felsted, Essex E. H. LOCKWOOD

34. Comment on class room Note 21.

Let the goal-posts be at $(a, 0)$ and $(-a, 0)$; the line l a perpendicular through $(x, 0)$; and P, corresponding to $\theta_{\max}$, at (x, y), then the locus of P as x varies is given by

$$y^2=(x+a)(x-a) \quad \text{or} \quad \frac{x^2}{a^2}-\frac{y^2}{a^2}=1.$$

The distance between the intersections of the curve and the asymptote, on the line l, is

$$x-y=x-\sqrt{(x^2-a^2)}=\frac{a^2}{x+\sqrt{(x^2-a^2)}}$$

which decreases as x increases.

In practice a player would probably prefer P to be at least 14 yd. from goal, so that we have $x>30$ ft., $a=9\frac{1}{4}$ ft. and $(x-y)<1\frac{1}{2}$ ft. which is small.

We may therefore suggest that a conversion is best taken from a point such that its distance from the point of touch-down is equal to, or about 1 ft. less than, the distance of the latter from the centre of goal. It is interesting to note that $(x-y)$ is the difference between the Arithmetic and Geometric Means of $(x+a)$ and $(x-a)$.

Technical College, Kingston-upon-Thames L. H. LE-BON

35. The hypotenuse number.

The quantity $\sqrt{a^2+b^2}$ occurs in a number of apparently unconnected topics in Analysis, such as:

1. The solution of the equation $a \cdot \cos\theta+b \cdot \sin\theta=c$.
2. The length of the perpendicular from a given point to the straight line $ax+by+c=0$.
3. The differentiation and integration of $e^{ax} \cdot \sin(bx)$.
4. The modulus of the complex number $a+ib$.

My pupils have found it convenient and helpful to have a name for $\sqrt{a^2+b^2}$, and, for obvious reasons, we call it " the hypotenuse number ".

Heath Grammar School, Halifax D. M. HALLOWES

36. On classroom note 21.

Whilst admiring the technique of allying mathematics to problems of the rugby field, the writer wishes to bring to notice the following points :

(i) The method breaks down for tries scored between or immediately behind the posts. It is easily shown that the points mentioned lie on the hyperbola

$$y^2 - x^2 = c^2.$$

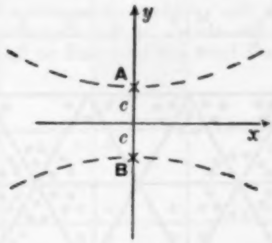


FIG. 1.

(ii) Besides kicking the ball behind the uprights, would not the kicker be influenced by the cross-bar? Therefore, would it not be more correct to consider the angle subtended by the cross-bar? In this case, it will be found that the optimum points lie on the hyperbola

$$y^2 - x^2 = c^2 + d^2$$

where d is the height of the cross-bar.

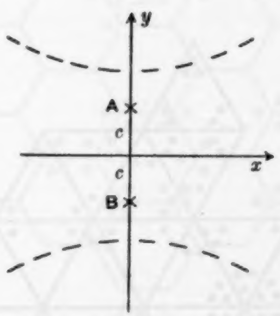


FIG. 2.

University of Leicester R. BUCKLEY

1926. There is no philosophy which is not founded on a knowledge of the phenomena, but to get any profit from this knowledge, it is absolutely necessary to be a mathematician.—Daniel Bernoulli, quoted by Clifford Truesdell. [Per Prof. W. G. Bickley.]

MATHEMATICAL NOTES

2829. Dominoes numbered in the corners.

Dominoes with three, four, or six sides have usually been constructed with the numbers attached to the edges, so that the condition for assembly is merely that every pair of contiguous edges must be identically numbered. (See, e.g., Macmahon, *New Mathematical Pastimes.*) If however the corners are numbered, and every corner of the set which surrounds each vertex must be identically numbered, the problem of assembly is much more difficult. Some simple assemblies are shown in the diagrams; those that are conspicuous by their absence I have not been able to do.

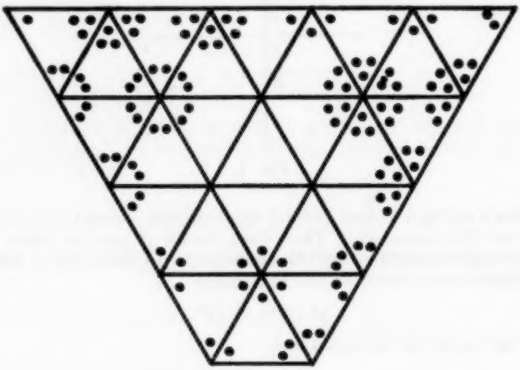


FIG. 1.

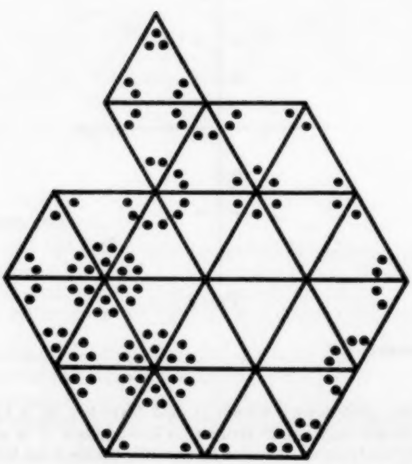


FIG. 2.

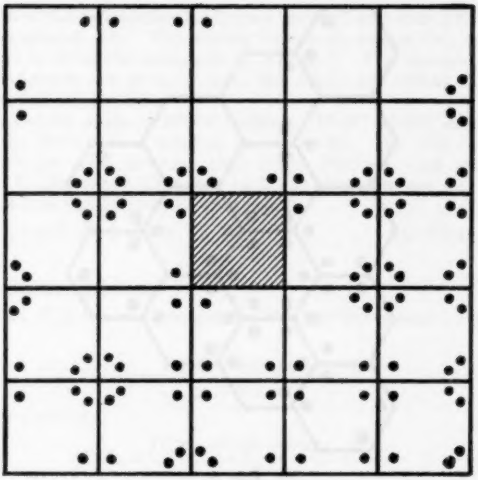


FIG. 3.

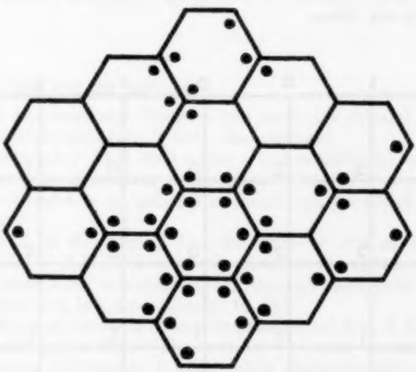


FIG. 4.

The numbers of dominoes in sets of this kind rise sharply. With six-sided dominoes the sets blank, blank-one, blank-one-two, . . . have 1, 14, 114, . . . pieces. The blank-one-two-three three-sided set has the same number (24) as the blank-one-two four-sided set.

The Editor tells me that he has constructed three-dimensional cubical dominoes, presumably with faces numbered (similar to dice). Perhaps someone would care to experiment with cubical dominoes numbered at the corners!

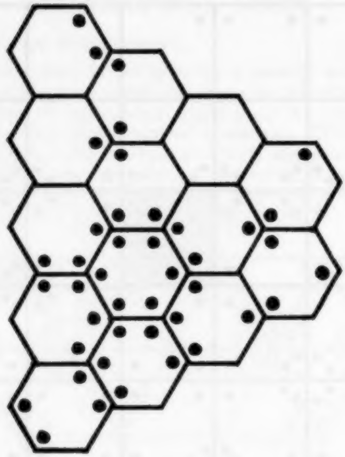


Fig. 5.

The more difficult variety of four-sided dominoes can be arranged in a rectangle four by six, thus

Fig. 6.

I have still been unable to arrange them three by eight, which seems to me *most* surprising as I would have expected this to be the easiest arrangement.

C. Dudley Langford

2830. Halves and thirds again.

A is the foot of the perpendicular from a point C to a line AB, and O is any point on the segment AB. The circles centre O, radius OC, and centre C, radius CB meet at D on the same side of AB as C. F is the mid-point of OB. The perpendiculars to AB at O, F meet the minor arc CD at E, G. Then, if $BC = 2AC$, arc $DG = 2$ arc GE.

Let $OC = 1$, and let angle $AOC = x$ radians. If $BC = 2AC$, angle $ABC = \pi/6$ radians, and so $BC = 2 \sin x$ whence arc $CD = 2x$. But $OB = 2 \sin (x - \pi/6)$ and so $FO = \sin (x - \pi/6)$, proving that angle $FGO = x - \pi/6$ radians and so arc $GE = x - \pi/6$. Since arc $ED = 2x - (\pi/2 - x)$ therefore arc $ED = 3$ arc GE.

I am indebted to the Editor for this proof.

Willesden, Holywood, N. Ireland ARCHIBALD H. FINLAY

2831. An examination question.

To prove that, if $f(x)$ is a polynomial *of the first degree* and $\alpha \neq \beta$, then for all values of x

$$f(x) = \frac{f(\beta)(x-\alpha) + f(\alpha)(\beta - x)}{\beta - \alpha}.$$

PROOF (*from a script*) :

$$\text{R. S.} = \frac{f(\beta x) - f(\alpha\beta) + f(\alpha\beta) - f(\alpha x)}{\beta - \alpha}$$

$$= \frac{f(\beta x) - f(\alpha x)}{\beta - \alpha} = \frac{f(x)(\beta - \alpha)}{\beta - \alpha}$$

$$= f(x) = \text{L. S.}$$

E. A. MAXWELL

2832. The Simson line and the cardioid.

The connexion between the Simson line and the deltoid is well known. Here is a problem that connects it with the cardioid.

Given a circle, a point P on it, and a line ; is it possible to inscribe a triangle in the circle so that the given line is the Simson line of P w.r.t. the triangle? The answer is that there is an infinity of solutions, provided that the line is " not too far away ".

Take any point A on the circle. Let the circle on PA as diameter cut the given line in M, N. If AM, AN meet the given circle again in C and B, then it is easily shown that ABC is a satisfactory triangle, and that if BC meets the given line at L, then PL is perpendicular to BC.

The condition for a solution is that it is possible to find A so that the circle on PA intersects the given line. The envelope of such circles, for various A, is the cardioid having its cusp at P and vertex diametrically across the given circle. The condition is that the given line must intersect this cardioid.

It is instructive to invert the figure w.r.t. P. The result is a like configuration, with the roles of the line LMN and the circle ABC interchanged. The inverse of the cardioid is the parabola, which is the envelope of lines through A, perpendicular to PA, as A moves along a given line.

University of Malaya RICHARD K. GUY

2833. Proofs that $2^{32}+1$ is composite.

Note 2816 shows that this fifth Fermat number has the factor 641.

This number has three simple connections with powers of 2 and 5. With modulus 641 we have

$$5\times 2^7 = 640 \equiv -1 \quad \dots\dots (1)$$
$$5^4 = 625 \equiv -2^4 \quad \dots\dots (2)$$
$$2^{11} = 2048 \equiv 125 = 5^3 \quad \dots\dots (3)$$

We can use these or combinations of them to prove the result.

From (1)	$5^4\times 2^{28}\equiv 1$
and so by (2)	$-2^{32}\equiv 1$
Again from (1)	$5^3\times 2^{21}\equiv -1$
Hence by (3)	$2^{11}\times 2^{21}\equiv -1$
i.e.	$2^{32}\equiv -1$
Alternatively, (2) gives	$2^{12}\equiv -5^{12}$
and so by (3)	$2^{12}\equiv -2^{44}$
and hence	$2^{32}\equiv -1$
As another alternative :	
(3) gives	$2^{33}\equiv 5^9$
	$\equiv 5\times 2^8$ by (2)
	$\equiv -2$ by (1).

Kraitchik (*Mathematical Recreations*, page 73) uses (1) and (2) to prove the result.

It is clear that $5^{32}\equiv 2^{32}\equiv -1$ so that 641 is also a factor of $5^{32}+1$. It follows readily that it is a factor of $10^{16}+1$.

Ingatestone, Essex B. E. LAWRENCE

2834. A walk in the rain [Note 2721].

In attempting the problem of finding the correct speed for a man so that he gets least wet, various assumptions may be made. In this note it is assumed that

(i) The wetting power of the rain is proportional to the *square* of the relative velocity.

(ii) The man's shape is that of a right block, height a, breadth b, depth (back to front) c.

The only case considered is when the rain falls vertically with velocity u and the man moves forward with velocity v.

Let $v = u\tan\theta$.

Square of relative velocity $= u^2\sec^2\theta$.

Exposed surface $= (a\sin\theta + c\cos\theta)b$.

Time for distance s $= \dfrac{s}{v} = \dfrac{s}{u}\cot\theta$.

The amount of wetting is proportional to

$$u^2\sec^2\theta(a\sin\theta + c\cos\theta)\frac{s}{u}\cot\theta$$

$$= us\left(\frac{a}{\cos\theta} + \frac{c}{\sin\theta}\right).$$

This will be a minimum if

$$\frac{a\sin\theta}{\cos^2\theta} - \frac{c\cos\theta}{\sin^2\theta} = 0$$

i.e. if
$$\tan^3\theta = \frac{c}{a}$$

or
$$v^3 = \frac{c}{a}u^3.$$

It is interesting to see that the *short fat man*, for whom c/a is greater than for the tall thin man, *must move faster* than the other.

C. O. TUCKEY

2835. A theorem on prime numbers.

The expression

$$E(x) = \lambda x + \sum_{k=1}^{\nu} a_k x^{r_k}$$

determines a natural number if the following conditions are satisfied :

$$r_k \quad (k = 1, 2, 3, \ldots, \nu) \text{ is prime. } \ldots\ldots\ldots\ldots\ldots (1)$$

$$a_k = 1/r_k \quad (k = 1, 2, 3, \ldots, \nu)\ ; \ldots\ldots\ldots\ldots\ldots (2)$$

$$\lambda = 1 - \sum_{k=1}^{\nu} a_k\ ; \ldots\ldots\ldots\ldots\ldots (3)$$

$$x \text{ is a natural number.} \ldots\ldots\ldots\ldots\ldots (4)$$

PROOF. For $x = 1$ the expression E becomes

$$\lambda + \sum_{k=1}^{\nu} a_k.$$

By condition (3) this has the value 1, which proves the theorem for $x = 1$. Assume that the theorem holds for $x = n$, and consider

$$\begin{aligned} E(n+1) &= \lambda(n+1) + \sum_{k=1}^{\nu} a_k (n+1)^{r_k} \\ &= \left\{\lambda n + \sum_{k=1}^{\nu} a_k n^{r_k}\right\} + \left\{\lambda + \sum_{k=1}^{\nu} a_k\right\} + \frac{1}{r_1} \sum_{k=1}^{r_1 - 1} \binom{r_1}{k} n^{r_1 - k} \\ &\quad + \frac{1}{r_2} \sum_{k=1}^{r_2 - 1} \binom{r_2}{k} n^{r_2 - k} + \ldots + \frac{1}{r_\nu} \sum_{k=1}^{r_\nu - 1} \binom{r_\nu}{k} n^{r_\nu - k}. \end{aligned}$$

By (3), and our hypothesis

$$\lambda + \sum_{k=1}^{\nu} a_k, \quad \lambda n + \sum_{k=1}^{\nu} a_k n^{r_k},$$

are integral.

By condition (1) the numbers

$$\binom{r_k}{s} \quad (s = 1, 2, \ldots, r_k - 1)$$

are divisible by r_k.

Consequently the expressions

$$\sum_{s=1}^{r_k - 1} \binom{r_k}{s} n^{r_k - s} \qquad (k = 1, 2, \ldots, \nu)$$

are divisible by r_k, which completes the proof by induction.

The theorem remains true if some r_k have the value unity.

Belgrade, Yougoslavie D. S. MITRINOVITCH

2836. Equivalence of two sets of inequalities.

If σ_k denotes the fundamental symmetrical function of real variables $x_1\, x_2, \ldots, x_n$ of order $k (\leqslant n)$, prove the equivalence of

$$\sigma_k > 0 \quad (k=1, 2, \ldots, n) \qquad \text{(1)}$$

and

$$x_k > 0 \quad (k=1, 2, \ldots, n). \qquad \text{(2)}$$

Solution. It is sufficient to prove that (1) implies (2). Now let us form the algebraic equation of degree n, the solutions of which are

$$x_1, x_2, \ldots, x_k.$$

The equation is of the form

$$x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} - \ldots + (-1)^n \sigma_n = 0.$$

Can this equation have any negative solution? Let us consider the equation

$$(-x)^n - \sigma_1(-x)^{n-1} + \sigma_2(-x)^{n-2} - \ldots + (-1)^n \sigma_n = 0,$$

or

$$x^n + \sigma_1 x^{n-1} + \sigma_2 x^{n-2} + \ldots + \sigma_n = 0.$$

The conditions (1) imply that the polynomial

$$x^n + \sigma_1 x^{n-1} + \sigma_2 x^{n-2} + \ldots + \sigma_n$$

is zero for no positive x; since the polynomial is zero for $x = x_k$, and x_k is real, we infer that

$$x_k > 0 \quad (k=1, 2, \ldots, n)$$

what was requested: (1) *implies* (2). D. S. Mitrinovitch

2837. Problème sur les progréssions arithmétiques.

Considérer les progréssions arithmétiques :

$$\begin{array}{llll} a_r, & a_r + \alpha_r, & \ldots, a_r + (n-1)\alpha_r & (r = 1, 2, \ldots, p); \\ b_s + (n-1)\beta_s, & b_s + (n-2)\beta_s, & \ldots, b_s & (s = 1, 2, \ldots, q). \end{array}$$

et former les produits suivants :

$$\prod_{r=1}^{p} a_r, \quad \prod_{r=1}^{p} (a_r + \alpha_r), \ldots\ldots, \quad \prod_{r=1}^{p} [a_r + (n-1)\alpha_r]; \qquad \text{(1)}$$

$$\prod_{s=1}^{q} [b_s + (n-1)\beta_s], \quad \prod_{s=1}^{q} [b_s + (n-2)\beta_s], \ldots\ldots, \quad \prod_{s=1}^{q} b_s. \qquad \text{(2)}$$

Trouver la somme des produits des termes respectifs de deux suites (1) et (2).

Remarque. La somme

$$\prod_{r=1}^{p} a_r + \prod_{r=1}^{p} (a_r + \alpha_r) + \ldots + \prod_{r=1}^{p} [a_r + (n-1)\alpha_r]$$

a déterminé *K. Miloševič* [*Bulletin de la Société des mathématiciens et physiciens de Serbie*, t.II, fasc.1–2 (1950), p. 69–74].

Université de Belgrade, Yugoslavia. D. S. Mitrinovitch

2838. Problème sur une équation fonctionelle.

Trouver les solutions continues de l'équation fonctionnelle

$$af(x) = f(x+b),$$

où a et b sont des constantes. D. S. Mitrinovitch

2839. The contour integral of a derivative.

In an introductory course on complex function theory it is convenient to have the theorem

$$\int_c f'(z)\,dz = f(b) - f(a)$$

for a contour integral along an arc C joining the points a, b available before Cauchy's theorem is proved, at the price of *assuming* the continuity of the derivative.

Let $f(z)$ have a continuous derivative in an open domain D and let C be a regular arc in D with end points a, b. Let the parametric equations of C be $x = x(t)$, $y = y(t)$ where $x(t)$, $y(t)$ have continuous derivatives for $t_0 \leqslant t \leqslant t_1$, and $x(t_0) + iy(t_0) = a$, $x(t_1) + iy(t_1) = b$.

If $f(z) = u(x, y) + iv(x, y)$ then

$$f'(z) = u_x + iv_x = v_y - iu_y$$

and so

$$\begin{aligned}\int_c f'(z)\,dz &= \int_{t_0}^{t_1} \{u_x[x(t), y(t)] + iv_x[x(t), y(t)]\}\{\dot{x}(t) + i\dot{y}(t)\}\,dt \\ &= \int_{t_0}^{t_1} (u_x\dot{x} - v_x\dot{y})\,dt + i\int_{t_0}^{t_1} (u_x\dot{y} + v_x\dot{x})\,dt.\end{aligned}$$

Writing

$$G(t) = u[x(t), y(t)], \qquad H(t) = v[x(t), y(t)]$$

we have

$$\begin{aligned}\frac{dG(t)}{dt} &= u_x\dot{x} + u_y\dot{y} = (u_x\dot{x} - v_x\dot{y}) \\ \frac{dH(t)}{dt} &= v_x\dot{x} + v_y\dot{y} = v_x\dot{x} + u_x\dot{y}\end{aligned}$$

and so

$$\int_c f'(z)\,dz = \Big[G(t) + iH(t)\Big]_{t_0}^{t_1} = f(b) - f(a)$$

Reading University. E. J. Ternouth

2840. Schur's inequality.

For $\mu = 1$ this is

$$S \equiv f(x, y, z\,;\,1) \equiv \Sigma x(x-y)(x-z) \geqslant 0$$

where x, y, z are positive.

Watson (Vol. 37 p. 244, Vol. 39 p. 207) has given two identities of the form $AS = B$, the first of total degree 5 and the second of total degree 4. Several other related identities have been given by Neville (Vol. 40 p. 216, Vol. 40 p. 288).

The following identity, again of degree 5, is neater than those given. It is also very easy to prove.

Put $l = y - z$, $m = z - x$, $n = x - y$; then $l + m + n = 0$.

Then
$$\begin{aligned}S &= -xmn - ynl - zlm \\ &= -xmn + yn(m+n) + zm(m+n) \\ &= yn^2 + zm^2 + (y + z - x)mn.\end{aligned}$$

Hence
$$l^2S = yn^2l^2 + zl^2m^2 + l^2mn(y + z - x)$$

and
$$\Sigma l^2S = 2\Sigma xm^2n^2 + lmn\Sigma l(y + z - x)$$

But $\quad \Sigma l(y+z-x) = \Sigma(y-z)(y+z-x) = \Sigma(y^2-z^2) - \Sigma x(y-z) = 0.$

Hence $\quad \Sigma l^2 S = 2\Sigma x m^2 n^2,$

i.e. $\quad \Sigma(y-z)^2 S = 2\Sigma x(x-y)^2(x-z)^2.$

University College, Bangor. C. C. H. BARKER

2841. On note 2776.

Is there really a query in note 2776? A. K. Rajagopal defines two interpretations of D^{-1} namely $\int_x^\infty \ldots dx$ and $\int_0^x \ldots dx$ and considers solutions of the Hermite and Associated Laguerre differential equations with $n = -1$. Distinguishing the two interpretations by suffices these solutions satisfy $y_1 + y_2 = \gamma$ where

(i) Hermite : $\gamma = -\frac{1}{2}\sqrt{\pi}e^{x^2}$.
(ii) Laguerre : $\gamma = \Gamma(\alpha)e^x x^{-\alpha}$.

These values of γ satisfy the respective differential equations so that if y_1 is a solution so is y_2 and there is no problem.

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2842. Great arcs and loxodromes.

Very few textbooks on spherical trigonometry give anything approaching a proof that the shortest path joining two points on the surface of a sphere lies along the great arc joining them, and beginning students are not always happy about accepting this result intuitively even if appeal is made to the stretching of an elastic string over the surface of a tennis-ball. The following demonstration, while not claimed as a perfectly rigorous proof, is nevertheless reasonably convincing and can be given at a very early stage in the development of the subject :

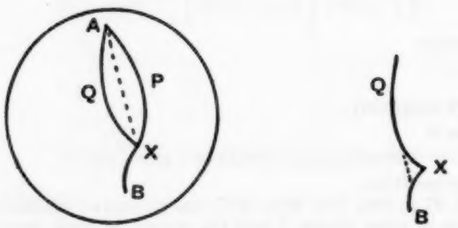

FIG. 1.

Let $APXB$ be the shortest curve joining the points A and B on the surface of a sphere, X being any point on this curve. Join AX by a great arc, and let AQX be the mirror image of APX in this great arc. The curve $AQXB$ then has the same length as $APXB$ but could obviously be shortened in the neighbourhood of the point X unless APX and AQX have the same direction at X, namely the direction of AX. Since this holds for any point X on the shortest curve AB it follows that this curve cannot cross any of the meridians drawn through A as pole, and hence must coincide with the meridian through B.

2. Another topic on which authors such as Todhunter and Leathem maintain a discreet silence is that of loxodromes or rhumb lines. When a class is shown that in general a great circle on the earth's surface cuts the circles of latitude at different angles someone will usually ask a question which can be rephrased in terms of " curves of constant bearing ", and it is useful to be able to pass on some information about such curves even if proofs are far beyond the mathematical understanding of the class.

Thus, from the theory of conformal transformations, it can be mentioned that the point on the earth's surface with geographical coordinates (θ radians East, ϕ radians North) is represented on Mercator's projection by the point with Cartesian coordinates (x, y) where $x=\theta$ and $y=\log_e(\sec\phi+\tan\phi)$, and that the angles between any two terrestrial curves intersecting at points other than the North and South Poles are preserved in the mapping by this projection. It follows that any straight line on Mercator's map represents a " curve of constant bearing " or loxodrome on the earth's surface. In particular, the line $y=x\tan\psi$ represents the loxodrome passing through the point of the Greenwich meridian which lies on the Equator, and having the bearing ψN of E. At any point on this curve we have $\log_e(\sec\phi+\tan\phi)=\theta\tan\psi$, and if $\psi\neq 0$ then θ becomes infinite as ϕ approaches $\pi/2$ from below. The curve therefore crosses all meridians of longitude infinitely often.

It is interesting to note also that the length of the loxodrome joining two points can easily be found by integration, for: Consider a three-dimensional Cartesian coordinate system with origin at the centre of the earth and with the x-, y- and z-axes passing through the points (ON, OE), $(ON, \frac{1}{2}\pi E)$ and the North Pole. Then $x=r\cos\phi\cos\theta$, $y=r\cos\phi\sin\theta$ and $z=r\sin\phi$, where r is the radius of the earth, and the parametric equations of the loxodrome considered above are

$$x=r\cos\phi\cos\left\{\frac{1}{\tan\psi}\log(\sec\phi+\tan\phi)\right\},$$

$$y=r\cos\phi\sin\left\{\frac{1}{\tan\psi}\log(\sec\phi+\tan\phi)\right\}$$

and $$z=r\sin\phi.$$

The length of arc of the loxodrome between the points with latitudes $\phi_1 N$ and $\phi_2 N$ is

$$\int_{\phi_1}^{\phi_2}\sqrt{\left(\frac{dx}{d\phi}\right)^2+\left(\frac{dy}{d\phi}\right)^2+\left(\frac{dz}{d\phi}\right)^2}\,d\phi.$$

The integrand of this expression simplifies to $r\operatorname{cosec}\psi$, giving the length of arc as $r(\phi_2-\phi_1)\operatorname{cosec}\psi$.

By considering loxodromes which map under Mercator's projection into parallel lines with slope $t=\tan\psi$ it is easily seen that the length of the loxodrome joining the points with coordinates $(\theta_1 E, \phi_1 N)$ and $(\theta_2 E, \phi_2 N)$ is

$$(\phi_2-\phi_1)\frac{r}{t}\sqrt{(t^2+1)}$$

where $$t(\theta_2-\theta_1)=\log_e\frac{\tan(\phi_2/2+\pi/4)}{\tan(\phi_1/2+\pi/4)}.$$

Finally it might be remarked, by taking $\phi_1=0$ and $\phi_2=\pi/2$, that a man travelling from the Equator in a constant direction ψN of E, $(\psi>0)$ will cover only a finite distance $\frac{1}{2}r\pi\operatorname{cosec}\psi$ before reaching the North Pole although he will have encircled the Pole infinitely many times if $\psi<\pi/2$.

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2843. A diophantine equation.

The general solution of the equation

$$p^2 - 2q^2 = -1$$

is given by

$$p_n = 3p_{n-1} + 4q_{n-1}, \quad q_n = 2p_{n-1} + 3q_{n-1}$$

with $p_1 = q_1 = 1$, or equally by

$$p_n = 6p_{n-1} - p_{n-2}, \quad q_n = 6q_{n-1} - q_{n-2}$$

with $p_1 = q_1 = 1$, $p_2 = 7$, $q_2 = 5$.

Since $(p_n + 2q_n)^2 = 2q_n q_{n+1} - 1$ it follows that a general solution in integers of the equation

$$\frac{x^2+1}{y^2+1} = \frac{y^2+1}{z^2+1}$$

is $\qquad x = p_n, \quad z = p_{n-1}, \quad y = \sqrt{(2q_n q_{n-1} - 1)} = p_{n-1} + 2q_{n-1}$

and another is

$$x = p_n, \quad z = p_{n+1}, \quad y = \sqrt{(2q_n q_{n+1} - 1)} = p_n + 2q_n.$$

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2844. A test for divisibility by 7.

1. Well known tests exist for divisibility by 3, 5, 9, 11 and 2^k $(k = 1, 2, \ldots)$. The following test for divisibility by 7 may also be of interest.

The number $a_n a_{n-1} \ldots a_1 a_0$ is divisible by 7 if and only if

$$S_7 = (a_0 - a_3 + a_6 - \ldots) + 3(a_1 - a_4 + a_7 - \ldots) + 2(a_2 - a_5 + a_8 - \ldots)$$

is divisible by 7.

The proof follows easily from the lemma:

If $10^j + t$ is divisible by 7 then so is

$$10^{j+3} - t, \ (j = 0, 1, 2, \ldots;\ t = 0, \pm 1, \pm 2, \ldots).$$

2. Since $10^2 + 1 = 101$, $10^3 - 1 = 37 \times 27$ and $10^3 + 1 = 13 \times 77$, it can easily be shown that, when p has the values 13, 37 or 101, then the corresponding values of S_p are:

$$S_{13} = (a_0 - a_3 + a_6 - \ldots) - 3(a_1 - a_4 + a_7 - \ldots) - 4(a_2 - a_5 + a_8 - \ldots);$$

$$S_{37} = [(a_0 + a_3 + a_6 + \ldots) - (a_2 + a_5 + a_8 + \ldots)] + 10[(a_1 + a_4 + a_7 + \ldots) - (a_2 + a_5 + a_8 + \ldots)];$$

$$S_{101} = (a_0 - a_2 + a_4 - \ldots) + 10(a_1 - a_3 + a_5 - \ldots).$$

Observe that S_7 and S_{13} differ only in the numerical coefficients of the second and third groups of digits.

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2845. Bisecting an area and its boundary.

In note 2808 J. G. Brennan showed that if D is a bounded plane convex region with boundary B there exists a straight line that bisects both B and the area of D. (Brennan supposes that B consists of a finite number of rectifiable arcs, but the boundary of a bounded convex set is a rectifiable curve in any case.) I suggest a somewhat different argument which shows that the convexity of D is irrelevant: if D is a Jordan region whose boundary is a rectifiable curve then there exists a straight line L that bisects the area of D and the length of B (in the sense that there is a segment of L such that the two arcs of

B determined by the end points of the segment have equal lengths). Let B be oriented. Take any point P on B; there is a line segment PQ, with Q on B, that bisects B. If this line segment does not bisect the area there is more area on one side than on the other; as P moves continuously on B, the area varies continuously (since there is choice of Q that makes the length PQ vary continuously: this ambiguity does not arise when D is convex). When P has reached the initial position of Q, right and left have been interchanged and so the amount of area on (say) the right of PQ, having initially been more than half the area of D and finally less, must at some point have been equal to half the area of D. The problem may be thought of as a limiting case of the proposition that there is a straight line bisecting any two plane areas (Courant and Robbins, *What is Mathematics?*, Oxford, 1941, p. 317).

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2846. Some identities.

A problem in a Sunday newspaper was set as follows:

" In my garden I had 2 square rose-beds of different sizes. I modified the layout making 2 square plots of equal area, but leaving the total area devoted to roses the same.

My 2 next door neighbours were also possessors of square rose-beds; each had 3, one of a certain size with 2 equal beds of another size, the dimensions being different for the 2 neighbours. They admired the change in my garden and followed suit. Each now has 3 square rose beds of equal size, but each has kept the same area as before devoted to roses. We have discovered that all our 8 rose-beds are now of the same size, also that the side of every rose-bed was and is an integral number of feet.

Given that the final beds are of the smallest possible size fulfilling the conditions, how large were our original beds?"

A little time shows that the solution follows from the results:

$$\left.\begin{aligned} 23^2+7^2&=2\,.\,17^2\\ 23^2+2\,.\,13^2&=3\,.\,17^2\\ 25^2+2\,.\,11^2&=3\,.\,17^2\end{aligned}\right\} \qquad \text{(i)}$$

It is perhaps interesting to generalise these results by omitting the " minimum " condition. It provides a diverting exercise in guesswork and verification.

A first stage is to get the identities:

$$[2p^2-(p\pm q)^2]^2+[2q^2-(p\pm q)^2]^2\equiv 2[p^2+q^2]^2 \qquad \text{(ii)}$$

$$[3r^2-2(r\pm s)^2]^2+2[3s^2-(r\pm s)^2]^2\equiv 3[r^2+2s^2]^2 \qquad \text{(iii)}$$

These show that it is sufficient for the final sides to be expressible as both p^2+q^2 and r^2+2s^2, for integral p, q, r, s.

Identities (ii) and (iii) are special cases of:

$$[(n+1)a^2-n(a\pm b)^2]^2+n[(n+1)b^2-(a\pm b)^2]^2\equiv(n+1)[a^2+nb^2]^2 \qquad \text{(iv)}$$

This further generalises to give:

$$m[(m+n)a^2-n(a\pm b)^2]^2+n[(m+n)b^2-m(a\pm b)^2]^2\equiv(m+n)[ma^2+nb^2]^2 \qquad \text{(v)}$$

This gives relations between sums of squares; and also gives the sequence of results:

$$\left.\begin{aligned} m[(m+n)a_0^2-nc_0^2]^2&=(m+n)a_1^2-nc_1^2\\ a_1&=ma_0^2+n(c_0-a_0)^2\\ c_1&=mc_0^2-(m+n)(c_0-a_0)^2\end{aligned}\right\} \qquad \text{(vi)}$$

so that if c_0/a_0 is a sufficiently close approximation to $\sqrt{[(m+n)/n]}$ then c_1/a_1 is a closer one.

Identity (v) can be more symmetrically stated as follows :

If, for some choice of the signs $\pm f \pm g \pm h = 0$ and if $l+m+n=0$ then

$$l(mh^2+ng^2)^2+m(nf^2+lh^2)^2+n(lg^2+mf^2)^2=0 \quad \text{...............(vii)}$$

This leads to the identity :

$$l(mh^2+ng^2)^2+m(nf^2+lh^2)^2+n(lg^2+mf^2)^2$$
$$\equiv(l+m+n)[mnf^4+nlg^4+lmh^4]+lmn(f+g+h)(-f+g+h)(f-g+h)(f+g-h) \quad \text{......(viii)}$$

School House, Sedbergh. J. H. DURRAN

2847. Diophantine quadratics.

The following very elementary method of dealing with certain Diophantine equations, including the Pellian equation in some cases, is unlikely to be new, but does not seem to be in the standard texts :

Example : $2x^2+7y^2=z^2$ can be written

$$2x^2-2y^2=z^2-9y^2$$

i.e.
$$2(x-y)(x+y)=(z-3y)(z+3y)$$

i.e.
$$\frac{2x-2y}{z-3y}=\frac{z+3y}{x+y}=\frac{m}{n} \quad \text{say,}$$

where m, n are integers. These give two homogeneous equations in x, y, z whose solution is $x:y:z=m^2-6mn+2n^2 : \; 2n^2-m^2 : 4mn-3m^2-6n^2$ and this gives the general solution in integers. G. A. GARREAU

2848. Representations of rational fractions.

The following results are the product of our efforts to express rational fractions as finite sums of distinct unit fractions (*i.e.* reciprocals of positive integers).

The first result of significance is that

$$\frac{1}{p}=\frac{1}{p+1}+\frac{1}{p(p+1)}$$

and this was used on simple particular cases

e.g.
$$\frac{2}{3}=\frac{1}{3}+\frac{1}{3}$$
$$=\frac{1}{3}+\frac{1}{4}+\frac{1}{3.4}$$
$$\frac{3}{5}=\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$$
$$=\frac{1}{5}+\frac{1}{6}+\frac{1}{5.6}+\frac{1}{6}+\frac{1}{5.6}$$
$$=\frac{1}{5}+\frac{1}{6}+\frac{1}{5.6}+\frac{1}{7}+\frac{1}{6.7}+\frac{1}{5.6+1}+\frac{1}{5.6(5.6+1)}$$

In itself this comprised an inductive process of representing any fraction as a sum of different unit fractions. However, on inspection of the denominators of the above expansion for $\frac{3}{5}$, a new pattern came to light.

It suggested that

$$\frac{3}{5}=\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{5.6}+\frac{1}{6.7}+\frac{1}{7.5}+\frac{1}{5.6.7},$$

which was soon verified.

By comparison with a polynomial whose zeros are $-\frac{1}{5}, -\frac{1}{6}, -\frac{1}{7}$;

$$\left(x+\frac{1}{5}\right)\left(x+\frac{1}{6}\right)\left(x+\frac{1}{7}\right)=x^3+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)x^2+\left(\frac{1}{5.6}+\frac{1}{6.7}+\frac{1}{7.5}\right)x+\frac{1}{5.6.7}$$

it was seen that

$$\tfrac{3}{5}=(1+\tfrac{1}{5})(1+\tfrac{1}{6})(1+\tfrac{1}{7})-1$$

and in general

$$\frac{p}{q}=\left(1+\frac{1}{q}\right)\left(1+\frac{1}{q+1}\right)\dots\left(1+\frac{1}{q+p-1}\right)-1$$

gave a useful device for finding the required representation of a rational fraction when reduced to its lowest terms.

Another proof is as follows:

there are positive integers p, q with p, q co-prime such that $qa-pb=1$......(1)

p, q are not unique since

$$a(q-\lambda b)-b(p-\lambda a)=1 \quad \text{........................(2)}$$

and from (2) we may assume $p<a$ in (1).

Then $$\frac{a}{b}=\frac{1}{bq}+\frac{p}{q}.$$

Now since p, q are co-prime we may repeat this process to obtain

$$\frac{p}{q}=\frac{1}{qq_1}+\frac{p_1}{q_1} \quad \text{with } p_1<p$$

Finally $$\frac{a}{b}=\frac{1}{bq}+\frac{1}{qq_1}+\frac{1}{q_1q_2}+\dots+\frac{1}{q_{n-1}q_n}+\frac{p_n}{q_n}$$

where the p_s are strictly decreasing

$$\therefore \quad p_n=1 \quad \text{for some } n=N \text{ say.}$$

This gives the required decomposition of $\dfrac{a}{b}$.

The lack of uniqueness leads us to propose the following problem:

Find distinct positive integers for each letter of the alphabet to satisfy the set of equations;

$$\begin{aligned}\frac{1}{a}&=\frac{1}{b}+\frac{1}{c}=\frac{1}{d}+\frac{1}{e}=\frac{1}{f}+\frac{1}{g}\\&=\frac{1}{h}+\frac{1}{i}+\frac{1}{j}=\frac{1}{k}+\frac{1}{l}+\frac{1}{m}\\&=\frac{1}{n}+\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+\frac{1}{s}+\frac{1}{t}\\&=\frac{1}{u}+\frac{1}{v}+\frac{1}{w}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\end{aligned}$$

21 *Fairview Crescent, Harrow.* B. Bolt and E. Wright

1927. It is a product of horrible imagination, of a mind that seems to be toppling over the edge of sanity, but which recovers itself at the last moment with so splendid an assurance that one feels like rising from one's seat to cheer. It is a mark of a great dramatist that he can count to infinity, and then add one. Mr. Williams does something even more miraculous; he counts to infinity, and then adds, not one, but two.—Harold Hobson in the *Sunday Times*, September 21, 1958.

so that if c_0/a_0 is a sufficiently close approximation to $\sqrt{[(m+n)/n]}$ then c_1/a_1 is a closer one.

Identity (v) can be more symmetrically stated as follows :

If, for some choice of the signs $\pm f \pm g \pm h = 0$ and if $l+m+n=0$ then

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where m, n are integers. These give two homogeneous equations in x, y, z whose solution is $x:y:z=m^2-6mn+2n^2 : 2n^2-m^2 : 4mn-3m^2-6n^2$ and this gives the general solution in integers. G. A. Garreau

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it was seen that

$$\tfrac{3}{5}=(1+\tfrac{1}{5})(1+\tfrac{1}{6})(1+\tfrac{1}{7})-1$$

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p, q are not unique since

$$a(q-\lambda b)-b(p-\lambda a)=1 \quad \dots\dots\dots\dots\dots\dots\dots(2)$$

and from (2) we may assume $p<a$ in (1).

Then
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Now since p, q are co-prime we may repeat this process to obtain

$$\frac{p}{q}=\frac{1}{qq_2}+\frac{p_1}{q_1} \quad \text{with } p_1<p$$

Finally
$$\frac{a}{b}=\frac{1}{bq}+\frac{1}{qq_1}+\frac{1}{q_1q_2}+\dots+\frac{1}{q_{n-1}q_n}+\frac{p_n}{q_n}$$

where the p_s are strictly decreasing

$$\therefore \quad p_n=1 \quad \text{for some } n=N \text{ say.}$$

This gives the required decomposition of $\dfrac{a}{b}$.

The lack of uniqueness leads us to propose the following problem:

Find distinct positive integers for each letter of the alphabet to satisfy the set of equations;

$$\begin{aligned}\frac{1}{a}&=\frac{1}{b}+\frac{1}{c}=\frac{1}{d}+\frac{1}{e}=\frac{1}{f}+\frac{1}{g}\\&=\frac{1}{h}+\frac{1}{i}+\frac{1}{j}=\frac{1}{k}+\frac{1}{l}+\frac{1}{m}\\&=\frac{1}{n}+\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+\frac{1}{s}+\frac{1}{t}\\&=\frac{1}{u}+\frac{1}{v}+\frac{1}{w}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\end{aligned}$$

21 *Fairview Crescent, Harrow.* B. Bolt and E. Wright

1927. It is a product of horrible imagination, of a mind that seems to be toppling over the edge of sanity, but which recovers itself at the last moment with so splendid an assurance that one feels like rising from one's seat to cheer. It is a mark of a great dramatist that he can count to infinity, and then add one. Mr. Williams does something even more miraculous; he counts to infinity, and then adds, not one, but two.—Harold Hobson in the *Sunday Times*, September 21, 1958.

REVIEWS

Basic Geometry. By Birkhoff and Beatley. (Chelsea Publishing Company, New York). Pp. 294.

This book is a reprint of the original edition published in 1940 which for some reason was not reviewed in the Gazette. Although difficult to use in this country, it is an important book. It is meant to cover a year's work (American style) and seems to be directed at the 15–16 age-group. The emphasis throughout is on logical structure and not on a catalogue of facts. Indeed the authors state as one of their main aims " to make (the student) critical of his own, and others', reasoning ; to see the need for assumptions, definitions, and undefined terms behind every body of logic ; to distinguish between good and bad arguments ; to see and state relations correctly and draw proper conclusions from them ". How far it succeeds in this is a matter of opinion. In the defined region of plane geometry it certainly does ; but it also introduces " real life " dilemmas and controversies geared to the teen-ager's interests where strict logical analysis is not possible. While the authors show the need of examining the assumptions underlying popular opinion in such questions, they leave to the teacher any instruction in the art of marshalling and weighing evidence and preferences. Since the problems include such questions as " Is it ever justifiable to lie to a person who is critically ill? " It is clear that this is going to be no light task. In fact, the book would be a constant stimulus in the hands of a good teacher ; in the hands of a bad teacher's class it might prove a prolific source of red herrings.

The geometrical development is based on five principles (assumptions), as follows : (1) Points on a line can be numbered so that number-differences measure distances. (2) The same for angles between half-lines at a point (3) Two points define a unique line. (4) Equality of all straight angles. (5) SAS case of similarity. The properties of the number-system are assumed in relation to (1) and (2) but listed (including Dedekind's Theorem) with great thoroughness in an appendix. From these five assumptions seven theorems are strictly derived : the two remaining cases of similarity, the isosceles triangle theorem and its converse, the angle-sum of a triangle, the equidistant property of the perpendicular bisector, the uniqueness of the perpendicular from a point to a line, and the Theorem of Pythagoras (by similarity, of course). Development now proceeds rapidly, via parallels (whose existence and uniqueness can now be proved), rectangular networks, circles, regular polygons, constructions, culminating in the pentagon-decagon construction. Area is now introduced, and its underlying assumptions laid bare ; the area of a circle is obtained as a limit. Continuous variation, inequalities, and loci round off the course, which touches briefly on power, coaxal circles, inversion and projection. The book concludes with a short section on reasoning and abstract logic, which could profitably be amplified considerably.

There are plenty of exercises, many of which are theorems in English books ; considerable guidance is given by the sequence of exercises to an intelligent pupil ; evidently all are expected to be worked. There are few " trivial riders ". But then, that is not the point, for the book could not be used in a normal English class. But it should be required reading for every teacher of Geometry, and a bright set would profit greatly by discussion of the underlying assumptions which it so skilfully lays bare.

Needless to say the production—which includes several photographs—is excellent, and the reviewer has noted only one error : the proof of Corollary 12c—that two sides of a triangle are greater than the third—does *not* apply to the second figure as is stated.

H. Martyn Cundy

Didactica Matematica Euristica. P. Puig Adam. (Instituto de Formacion del Profesorado de Ensenanza Laboral, Madrid). Pp. 136. Price 50 pesetas.

P. Puig Adam is one of the most original mathematics teachers of our time. This book describes thirty of his lessons on themes from arithmetic, algebra and geometry and on mathematical models taken from real life. The book is one of great strength. Originality is never sought for its own sake, a model is never shown off in order to be clever, " practical " methods of instruction are not employed if an abstract discussion achieves the objective more easily ; but in each lesson he goes straight to the point at issue and grapples with the difficulties. The lessons are straight from the classroom, the size and age of the class is usually stated, and each lesson is an experiment in which the author has participated rather than a lecture at which he has dispensed information.

Puig Adam is one of those who aim at a method of instruction in which the pupils have to convince themselves rather than be convinced by the master ; but he has no single method of achieving this. Often the class is divided into groups which work on number games. If operations are part of a game their abstract nature far from being a source of bewilderment to the class is a source of delight. The author can enthral an audience of mathematicians in a hotel lounge with a skilful display of conjuring. He can enthral a class in the same manner, but the tricks he does with a class all have a mathematical basis and display some idea clearly.

The same is true of his apparatus. " We live surrounded by mathematical models without noticing them " he says. Accordingly a parasol, string and cardboard are valued for their mathematical properties. A window catch displays symmetry operations in space, cards of press-studs give a lucid explanation of the " long " method of extracting a square root, *broken* models of regular solids in the school's store cupboards teach more than models in mint condition, and a child's toy gives a lesson on the parabola. It requires an eye of genius to see an integral domain in a box of bricks, but Puig Adam has this eye. If he is giving a simple lesson on an arithmetic progression he gives it in full knowledge that he is within easy reach of an advanced lesson on residue classes and rings. It is this assured confidence of working over the whole field of mathematics and knowing where he is going that gives the book its strength. From school lessons, perhaps on old and well-tried topics, fresh discoveries arise to charm us all.

But with all of this the traditional values are never lost—the aim is rather to approach them in a new way. The puzzles lead to a spontaneous desire for clear and precise explanation, and the physical models lead to abstract ideas. In good elementary exposition there is an artistic element of neatness and economy which makes the teaching of long established material a creative art every bit as much as the discovery of new theorems. The author is a master of this art, and the dedication of his book " to the joy of the pupils and to all those who come to love mathematics by inventing it " is no empty form of words ; so brush up your Spanish and read it.

T. J. Fletcher

Direct Mathematics. (Books I to IV and Teacher's Book). Pp. 120, 136, 135, 136 and 154. 1955 (Nelson).

This is a series of four books and a teacher's book for the Secondary Modern School. The author has planned the series for use by all streams, bright, average and slow, and has outlined the course for each stream ; less able pupils following the same main course but omitting certain chapters and proceeding at a slower rate.

The books contain arithmetic, algebra, geometry and a chapter on trigonometry. The chapters on arithmetic and mensuration are good ; the intro-

ductions are clear and well explained, examples are well graded and the work consistently referred to realistic situations both by description and by use. Stress is laid on the use of ready reckoners and the value of an approximate answer. The treatment of algebra and geometry is less happy. One feels that they have been introduced without full conviction. The geometry is very largely abstract and formal in its treatment except for a few problems in the progress tests and two chapters on field work. Little reference is made to geometry in the world around and this is particularly surprising in view of many such references in the arithmetic sections. The algebra is introduced much too quickly ; the emphasis is almost entirely on manipulation, the work is not used to clarify the pupil's ideas of arithmetical processes, formulation is omitted. The chapter on trigonometry introduces the tangent, sine and cosine ratios together then gives a set of problems on each. A treatment of the tangent only would have perhaps been more satisfactory ; one wonders whether the Secondary Modern pupil would ever disentangle the three ratios subsequently even with the aid of the given mnemonic.

A teacher who is able to supplement the geometry and algebra sections would find the series valuable. The teacher's book contains 31 very useful pages including historical notes on measures, suggestions for visits, suggestions for lecturettes, and a model lesson on the working of a clock. There are many photographs showing people using mathematics in varied occupations and a very attractive cover is made by the merging of these.

K. Sowden

Elementary Calculations. Book One. Pp. 112. 5s. 1955. Book Three. Pp. 123. 5s. 1956. (Harrap)

To quote from the preface to the series : " This present series has been designed for those who are likely to receive all their post-primary education at Secondary Modern Schools. Accordingly, the amount of explanation has been kept to a minimum and more emphasis has been laid on a carefully graduated series of exercises which the children can really do, leading up to practical applications in daily life ". There seems to be a suggestion here that less able children should have more practice in computation with less explanation and on inspection of Book One this would seem to be the idea behind its plan. The book bears a strong resemblance to Book One of Mr. Ward Hill's popular series *Mathematics for Modern Schools*. Many of the problems and mechanical examples are here again with the addition of some easier ones, but the work covered is about two-thirds, decimal fractions, multiplication and division of vulgar fractions being among the topics omitted. Other omissions, however, such as Roman numerals and the valuable introduction to addition of vulgar fractions are to be deplored. It is not easy to understand why these exercises, with their more mathematical content and their value in developing mathematical understanding should have been discarded in favour of long multiplication and long division of money. (Must we still inflict these tedious calculations on the less able pupil?) The problem exercises are realistic and of interest to children, and as a source of applications this is a valuable book to anyone who has not already a copy of *Mathematics for Modern Schools*, Book One.

Book Three is a much more satisfactory book. It includes, as new topics, averages, percentages, approximation, volumes and some work on areas of triangles and circles. The second half is called " Using Our Skills " and gives a well selected series of exercises on the usual arithmetic of citizenship (in which the pupil is allowed to use ready reckoners for long multiplication and long division of money).

The books are attractively presented, clear in print and diagram.

K. Sowden

Oxford Graded Arithmetic Practice. (Book Seven: Weights and Measures). By D. A. HOLLAND. Pp. 64. 2s. 6d. With Answers 3s. 6d. 1957. (Oxford University Press).

Books Five and Six were reviewed in detail in the *Mathematical Gazette* for December 1957. Book Seven follows a similar plan in providing carefully graded examples for practice in computation using the measures of length, weight, capacity and time.

K. SOWDEN

The Four Rules of Measurement. By K. A. HESSE. Pp. 62. 2s. 6d. Teacher's Edition. Pp. 70. 6s. 1958. (Longmans, Green and Co. Ltd.)

The avowed purposes of this book are to give ample practice in computation using the measures of length, weight, capacity and time and to aid the teacher to discover and remedy the pupils' weaknesses in these processes. A feature of the book is the " check pages ", one for each process. The pupil works one line of the relevant check page and if mistakes are made is directed by a note at the end of that line to the page giving the necessary practice. This practice is not merely repetition of the same type of example but is directed towards finding the exact difficulty, by including examples of the calculations which are intermediate stages in the process. For example, errors in the addition of pounds and ounces may be caused by confusion over the relationship between pounds and ounces or by lack of familiarity with the number 16, and exercises involving these two concepts are given separately. The emphasis throughout is on the necessity for pupils to have sufficient time to progress at their own rates.

The book is large, $9\frac{3}{4}$ inches by $7\frac{1}{4}$ inches and very pleasing in the variety of its lay-out and the clarity of its print. Answers are given in red below the examples, a considerable aid to the teacher when marking.

K. SOWDEN

Calculus. By E. S. SMITH, M. SALKOVER, and H. K. JUSTICE. (2nd edition). Pp. xiii, 520. 52s. 1958. (John Wiley, New York; Chapman & Hall)

The first edition of this book was published in 1938. This edition, the second, differs little from the same authors' *Unified Calculus*, which was published in 1947 and reviewed in the *Mathematical Gazette* in October, 1948. The sections on centroids, moments of inertia, and inflexions, have been expanded and transferred, with advantage, to different places in the book. A well illustrated chapter on Solid Analytical Geometry replaces the list of formulae in the earlier book; it deals with direction cosines, and the equations of the line, the plane, and quadrics referred to principal axes, for use in the chapter on multiple integrals. The examples are more numerous than in the earlier book, and the answers to the odd-numbered ones are given at the end of the text. There are no other changes of importance, but there are a few small ones, including the addition of a footnote to the proof of the formula for the derivative of a function of a function, which recognises the incompleteness of the proof and gives a reference to another book for a fuller treatment. Apart from its remark on this proof, the whole of the review of *Unified Calculus* is relevant to the present book, and the following comments may be read in conjunction with it.

The book begins with a chapter on limits, develops differentiation and integration together, and concludes with chapters on partial differentiation, multiple integrals and differential equations. The treatment throughout is clear and straightforward. The numerous diagrams are especially good, and important formulae are displayed in bold type. The emphasis of the book is on fundamental ideas and their applications to simple problems, rather than

on completeness and the more difficult theoretical considerations. Thus some topics, such as limits of the form $\lim_{x\to\infty} x^n/e^x$, the harder standard integrals, reduction formulae, the parallel axis theorem for moments of inertia, of which some discussion in the main text might be expected in a book of this size and scope, appear only in the worked examples, the exercises, or the table of integrals at the end. A longer chapter on numerical integration, and some discussion of change of order of integration in a repeated integral, would certainly be welcome. On the other hand, the chapter on curve tracing, and the second of the two chapters on limits, contain some rather difficult material for a book of this scope, and these chapters, together with the section on the relation between the hyperbolic functions and the rectangular hyperbola, could well be shorter. Occasionally a more difficult proof is given than is appropriate. Thus, the statement that two functions with the same derivative differ by a constant is made as soon as the concept of integration is introduced in Chapter 3, but for a proof of this the reader is referred to an example on Cauchy's Mean Value Theorem in Chapter 12.

When the differentiation of the elementary transcendental functions is introduced, the argument of the functions is taken to be a differentiable function u of x, and the results are all first obtained in the form exemplified by the equation $\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$. The introduction of u in this way seems an unnecessary complication at this point. The argument of the trigonometric functions is described as " an angle expressed in radians "; the distinction between an angle and its measure in radians is not very clearly brought out, and $\frac{d}{dx}\sin x^\circ$ is not evaluated. The many-valued nature of the inverse trigonometric functions is discussed, but the principal value is not defined. Hyperbolic functions are introduced rather late in the book, though when they have been introduced, the advantage of a hyperbolic over a trigonometric substitution in integrals involving $\sqrt{(x^2 \pm a^2)}$ is pointed out. Pedagogically helpful remarks are also made on the process of integration by parts, on the need for care with discontinuous integrands, and on the use of L'Hôpital's rule. There is a good chapter on Taylor's theorem and applications. The final chapter on differential equations deals mainly with linear equations with constant coefficients, and includes a discussion of damped oscillations. In its treatment of the harder cases in the examples rather than in the main text, and in its emphasis upon applications, this chapter epitomises the approach of the authors to their subject matter throughout.

A. G. Vosper

Numerical Trigonometry. By R. Walker. Pp. 184. 8s. 6d. 1957. (Harrap)

This is a good introductory book covering all the work for Mathematics at O Level. In the earlier chapters no knowledge of logarithms is demanded, so that it is possible to use this book to introduce Trigonometry very early. After an introductory chapter on Similar Figures and Bearings, chapters are devoted to sine, cosine, and tangent each containing the definition of the ratio, a discussion of the table and its accuracy and a graph for the first quadrant except for cosine. These chapters are independent of each other so that they can be taken in the order preferred by the individual teacher, and each contains a plentiful supply of well graded examples. A set of 10 revision papers follows this section.

Graphical solutions of trigonometrical equations are discussed, and the ratios for 30°, 45°, 60° after which logarithms are introduced. An excellent

chapter on three dimensional work follows, including a section on latitude and longitude, after which ratios for obtuse angles, sine rule and cosine rule are covered. A chapter on Navigation Problems will arouse interest especially among boys, and the book concludes with some miscellaneous topics such as triangle formulae, addition formulae and identities which will be useful for pupils taking Additional Mathematics. Examples are plentiful throughout, and there are several sets of revision exercises. Logarithmic, Trigonometrical, Square and Square Root tables are included and answers are supplied.

F. E. Chettle

Plane Trigonometry. By E. R. Heineman. 2nd Ed. Pp. 167. 24s. 6d. 1956. (McGraw-Hill)

This book covers a great deal of ground, at times rather sketchily, and it is difficult to see how it would fit into the normal scheme of work in a school in this country. The author begins with a discussion of directed segments and the rectangular co-ordinate system followed by definitions of the six trigonometrical ratios for general angles, then turns in the second chapter to acute angles with special mention of 30°, 60°, 90° and the type of problem based on solving right-angled triangles common in O level work. Identities based on the relationship $\sin^2\theta+\cos^2\theta=1$ follow with clear worked examples and plenty of exercises for the student including some of the type "if the statement is true, cite proof; if false, correct it" which are very good.

Methods of expressing functions of larger angles in terms of acute angles, radian measure, graphs of all trigonometrical functions, compound angle formulae with proofs for acute only, and product to sum formulae are all discussed and a short chapter on trigonometrical equations with solutions restricted to the 0–360° range is included. A section on logarithmic theory and the use of logarithms in computation is given, the American notation being used in place of the bar notation for negative characteristics, and this is followed by solution of triangles and inverse functions. The final chapter of 9 pages covers complex number up to and including Demoivre's Theorem.

The examples throughout are plentiful and interesting and the diagrams are very good. A student requiring a refresher course after a break from Mathematics would find a book such as this helpful.

F. E. Chettle

Mechanics. By H. Weiler. Pp. 521. 1957. (Pitman).

This book is intended for Intermediate Examinations in Universities, and also covers much of the work required for Scholarship candidates. There are however omissions, as will be pointed out, that make it insufficient in some respects for the latter. But the form and arrangement of the book is scholarly, and should give inspiration to the thoughtful student.

The order provided is logical, and at no stage is anything assumed that has not been led up to earlier in the book. The author sets out to provide an ordered presentation of Newtonian Mechanics, and lays stress on clear definitions, and statements of essential assumptions. He then develops the theory by theorems which are proved carefully. There are in some chapters rather many theorems, e.g. 15 in the chapter on Forces acting on a rigid body, and 16 on the Dynamics of a rigid body. Hence some less important results receive the same emphasis as vital results, so that the latter are not always sufficiently outstanding.

The other special feature of the book is the arrangement of examples. There are a number of examples through each chapter, many worked out but some left for the student. Then at the end of each chapter are two sets. The first provides questions on the bookwork of the chapter, labelled revision

questions, enabling the student to check that he has understood all the bookwork. The second set consists of problems on the chapter, divided into different types by lines. These look reasonably plentiful, but may prove rather short on some topics. There are no general miscellaneous examples.

The first nine chapters are concerned with the statics, kinematics and dynamics of a particle. Vector algebra is used, only addition and subtraction, and velocity in a plane is defined by the differentiation of a position vector. An opportunity is missed here of emphasising that all velocity is relative to a given origin, since the term actual velocity is used, rather than velocity relative to the earth. Projectiles and motion in a circle are treated thoroughly, but the enveloping parabola is not used, nor does the book include general motion on a curve, or radial and transverse components of acceleration and motion in a plane under the action of a central force.

Chapters 10–12 deal with forces on a rigid body, centre of gravity and Statics. The theory of the reduction of forces is set out very clearly with diagrams, starting with good three-dimensional figures illustrating the moment of a force about an axis. The statics includes a section on Shearing force and Bending moment, but does not examine the catenary, or introduce the method of Virtual Work, or the potential energy tests for equilibrium.

Chapter 13 is a long one on the dynamics of a rigid body, the theory being introduced carefully by considering the body to be made up of the sum of a finite number of particles. Motion about a fixed axis is discussed fully, including the reactions on the axis, and then the general motion of the body in a plane is discussed, though there are rather few examples on this section.

The book finishes with a chapter on Hydrostatics, including all that is necessary at this stage.

The book must appeal to the scholarly mind, by its insistence on a logical development, though it is so arranged that anyone can read the book in any order, but see how each section fits into the order given. There is rather more theoretical bookwork than most schoolboys like, but on the other hand there are a great number of worked examples which are always helpful. For a book of this standard it is a pity that the items mentioned above have been omitted, but a reader with a scholarly mind cannot fail to be stimulated to thought by the clear and logical development of the subject.

K. S. SNELL

Linear Equations. By P. M. COHN. Pp. 74. 5s. 1958

Sequence and Series. By J. A. GREEN. Pp. 78. 5s. 1958

Differential Calculus. By P. J. HILTON. Pp. 56. 5s. 1958

Elementary Differential Equations and Operators. By G. E. H. REUTER. Pp. 67. 5s. 1958. (Routledge and Kegan Paul).

These four little books introduce a new series of elementary texts edited by W. Ledermann. The series is addressed to the science student who is not specialising in mathematics. The intention is that what is proved shall be well proved and what is left unproved shall be clearly stated. The editor and authors are all members of the Mathematics Department of Manchester University. The volume on differential equations presents an interesting integral operator treatment independent of the Laplace transform, but the familiar operational use of D is not discussed. Apart from a study of remainders the account of series has no novel feature, unless one counts the very careful definitions as a novelty in an elementary text. The differential calculus volume is rather disappointing. We are told that $\sin\theta$ is a function of the angle θ measured in radians, so that presumably the value of $\sin^{-1}1$ is $\frac{\pi}{2}$

radians and not just a number; the reader is warned against taking the concept of arc length for granted, but on the next page area is taken for granted in the proof of the inequality $\sin x < x < \tan x$. The statement of the mean value theorem inadequately reveals its nature as an existence theorem, and we are not told the range of values of x for which $f(x)$ is required to be differentiable. I have left to last the volume on Linear Equations, to be able to end on a note of unqualified praise. This is a piece of presentation and exposition of the very highest order, which should be compulsory reading for every student in the first year of a mathematics degree course.

R. L. GOODSTEIN

Vector Analysis. By L. BRAND. Pp. xiii, 282. 35s. 1957. (New York: John Wiley and Sons; London: Chapman and Hall)

This book was " designed as a short course to give a beginning student the tools of vector algebra and calculus and a brief glimpse beyond into their manifold applications ". The rigour of the analytical treatment will particularly appeal to mathematicians, though the book is intended for use by engineers and physicists also.

The first three chapters, on free vectors, line vectors, and the differentiation of vector functions of one variable, are competently written. They contain numerous geometrical applications, and a wealth of other material which will be found useful by the student. The reviewer noted with interest a neat and, as far as he was aware, novel proof of the distributive property of vector products. This depends on deriving $\mathbf{u} \times \mathbf{v}$ from $\mathbf{v}$ by a succession of elementary operations, each of which is distributive.

Gradient, divergence and curl form the subject matter of the fourth chapter. (Unfortunately, " rot " is used instead of " curl ". How this can be " in the interests of a uniform notation ", as the author claims, is difficult to see.) However, in the reviewer's opinion, the treatment here is not satisfactory. After dealing with the gradient of a scalar in the normal way, the writer considers the gradient of a vector, and so is led to dyads and dyadics. Four pages are now devoted to a compact survey of the properties of dyadics, including their invariants. It is unlikely that a student meeting the subject in these pages for the first time would gain any understanding of it. He would certainly have no idea of what a dyadic really is: it is merely defined as a sum of dyads (ordered vector pairs). Incidentally, the author does not explicitly define the products $\mathbf{u} \cdot \mathbf{P}$ and $\mathbf{P} \cdot \mathbf{u}$ of a vector $\mathbf{u}$ and dyadic $\mathbf{P}$, nor, more serious, does he prove that $(\mathbf{u} \cdot \mathbf{P}) \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{P} \cdot \mathbf{v})$, justifying the notation $\mathbf{u} \cdot \mathbf{P} \cdot \mathbf{v}$.

The divergence and curl of a vector are defined as the scalar and vector invariants of its gradient. This leads readily to the familiar properties of these entities, but again the student cannot appreciate their physical significance through this treatment. This is surely *not* the way to introduce divergence and curl to a " beginning student ". It belongs to a text on Tensors. In his Preface, the author attempts to justify his treatment by saying that " the world is full of important tensors that the earnest student might just as well encounter head on ". This is an astonishing statement. To be taught about tensors while meeting vector algebra and calculus for the first time is to be taught to run when beginning to walk.

The fifth chapter contains an excellent account of the Integral Theorems, including many interesting topics, but strangely fails to mention that the divergence of a vector at any point is the limit of the flux per unit volume. This, an immediate consequence of Gauss' Theorem, is surely the real physical meaning of divergence, yet is nowhere mentioned in this book.

Of the following three chapters, on applications to dynamics, fluid mechanics

and electrodynamics, the second and third seem adequate—but that on dynamics is disappointingly meagre. It ends abruptly with a section on a body revolving about a fixed axis. There is no mention of rotating axes, no treatment of the general motion of a rigid body. It is a poor excuse (given in the Preface) for the latter omission that " the use of dyadics " in the book " must be held to a drastic minimum " (sic!), so " rigid dynamics (with its inertia dyadic) . . . have been regretfully omitted ". Does the author really think that a satisfactory vector treatment of rigid dynamics cannot be given without using the inertia dyadic?

The final chapter, on Vector Spaces, is a novelty in a book of this kind, which will be found readable, interesting and useful by mathematical students, though possibly not by engineers and physicists. The chapter is quite ambitious : it deals, among other things, with linear dependence, the Gramian determinant, reciprocal bases, Hilbert space, least-square approximations, closed and complete bases, and solution of linear equations : a considerable accomplishment in a mere 26 pages.

A few minor errors were noted : on p. 31, the term " basis " is used before its definition, on p. 37 ; in eqn. (11) on p. 147 the difference of the two sides is irrotational, not necessarily zero as stated ; on p. 36, bottom line, for " $4A$ " read " $4A^2$ " ; on p. 41, top line, for " basis **a**, **b**, **c** " read " basis reciprocal to **a**, **b**, **c** " ; on p. 73, in example 3, the parameter is v, not u.

The production and printing of the book are excellent.

A. Talbot

Algebra. By J. W. Archbold. Pp. xix, 440. 45s. 1958. (Pitman)

In recent years, the B.A., B.Sc. General and B.Sc. Special Degree examinations of the University of London have demanded some knowledge of modern ideas in linear algebra. It will therefore be a great convenience for the student to have in a single volume all the algebra now required for these examinations, expounded by an experienced member of the mathematics department of University College, London.

The book falls into two main parts. In the first part (Chs. 1–15) we have an account of the older material : inequalities, complex numbers, polynomials, theory of equations. But the treatment is fresh and forward-looking : the principle of induction is carefully explained and fully employed, preliminary ideas about ground fields are displayed, and particular attention is given to the factorisation properties of polynomials, and to the theory of rational functions and their partial fractions. All this is so well done that it is almost ungenerous to make one small criticism, that the teacher may well find that the book does little to encourage a class to an intelligent manipulative technique when dealing with partial fractions. Here the Association's Report on VIth Form algebra should be consulted.

In the second part of the book (Chs. 16–25) the student begins to grapple with modern ideas. Through simple geometrical examples, he is led to the abstract concepts of groups, rings and fields, and then on to the central themes of modern linear algebra. The treatment is abstract, but the ground field is restricted to be commutative and without characteristic ; the reader can therefore, if he wishes, continue to dwell in the familiar complex number field. Though there is a chapter on symmetric groups and one on determinants of order n, the main emphasis is on vector spaces and matrices, as far as the Cayley-Hamilton theorem. The author is quite right, to my mind, in dealing with the general ideas of matrix theory first, in some detail, before distracting the reader's attention by bringing in special types of matrix ; these, orthogonal, symmetric, skew-symmetric, Hermitian, are then discussed in a separate

chapter. From a geometer of Dr. Archbold's calibre, we might expect a slightly stronger geometrical flavour in this chapter and in the next two, on quadratic forms and on discriminants and resultants; but this might have been inconsistent with brevity, and after all there is an abundance of good texts on algebraic geometry.

Finally, very high praise must be given to the magnificent collection of exercises for the student: there are more than any single reader will need, and yet it is difficult to suggest sacrifices. There are plenty of good examination questions, but, particularly in the later sections, there are many good examples just short of the "theorem" standard, which have been carefully selected from the rich field of the "minor" periodicals; the *American Mathematical Monthly* and the *Gazette* have evidently been skilfully and assiduously ransacked. Altogether, the book is not only admirably suited to its particular purpose, it caters carefully for the examinee without restricting his horizons.

T. A. A. BROADBENT

The Enjoyment of Mathematics. By H. RADEMACHER and O. TOEPLITZ. Pp. 204. 36s. 1957. (Princeton University Press. London: Oxford University Press)

This is a translation (from the second edition of *Von Zahlen und Figuren* originally published in 1933) by H. Zuckermann who has also contributed two short new sections to the book.

A feature of the book is the number of complete and detailed proofs which it contains: written for the amateur mathematician it demands of him no previous knowledge of mathematics but a willingness to concentrate hard and work patiently through comparatively close arguments. Some of the material is of course to be found in the many popularisations of mathematics written since the first edition of this work appeared, but some things are still, I think to be found here alone; for instance the proof that no plane or spherical map requires more than five colours, and the proof that every finite plane set of points of span d may be enclosed in a circle of radius $d/\sqrt{3}$.

The translator sums up the spirit of the book by saying that it shows how mathematics can build a real and meaningful structure on a small foundation; while this is certainly true I think it also exhibits mathematics as the very language of reason.

The translation is excellent, being fresh and vigorous without being unfaithful to the original. This is a book which every school boy intending to read mathematics at the University should be encouraged to study.

R. L. GOODSTEIN

Fun with Geometry. By MAE and IRA FREEMAN. 62 pp. 10s. 6d. 1958. (Edmund Ward, London)

This attractively produced book is American in origin, but at least one figure and the letterpress has been anglicized. Every odd-numbered page from page 7 onwards is occupied by half-tone diagrams which are explained in the letterpress on the facing pages. Though the book is printed by offset or lithograph on ordinary paper the diagrams are beautifully clear and really illustrate the various topics. These are everyday examples of simple geometrical facts, including angles, similar triangles, circle, ellipse and parabola, spirals and helix, Lissajous' ("quiver") figures, Möbius strips, knots, tangrams and the regular convex polyhedra. A bright 12–14 year-old would enjoy this book, which would make an excellent school prize. A capable teacher could use it as a jumping-off point for many interesting projects. It would of course be pedantic to expect precision in a book of this kind, but two

statements do seem to be misleading : (i) a ball does not bounce off a wall at the same angle as it strikes it ; and (ii) there are far more than five solids whose faces are " flat and all alike ". But the last word must be one of praise for an attractive and stimulating production.

H. Martyn Cundy

Fantasia Mathematica. Edited by Clifton Fadiman. Pp. 298. $4.95. 1958. (Simon and Schuster. New York)

The author asks mathematicians to stay away from this collection of stories and verses with vaguely mathematical themes, but in fact none of them is more far fetched than Dunne's Experiment with Time which some mathematicians are said to have been misguided enough to take seriously.

Pride of place in the collection is rightly given to Aldous Huxley's beautiful story " Young Archimedes ". The Möbius band is featured in several stories and a möbius band island is the welcome excuse for an account of the four colour problem. Even the devil fails to solve Fermat's last theorem. There are two references to the *Mathematical Gazette*, but the Fable on page 294 is not on page 308 of my copy of the 1954 Gazette as stated.

R. L. Goodstein

The Tree of Mathematics. Edited by Glenn James. Pp. 420. $6. 1957. (The Digest Press, 14068 Van Nuys Blvd., Pacoima, California)

This collection of twenty-seven articles on a wide range of mathematical topics is a very welcome new venture in mathematical exposition. Amongst the subjects treated are abstract algebra, topology and theory of games, and contributors include E. T. Bell, H. S. M. Coxeter, M. Fréchet and Olga Tausky. The articles, naturally, differ both in their intention and design ; some cover a small section in considerable detail and others aim at a more general survey. Many are remarkably successful. It might be doubted whether the first few elementary articles should have been included in the volume—certainly they will not interest anyone able to read the more advanced articles. The Editor is to be warmly congratulated on a courageous and imaginative enterprise.

R. L. G.

A History of Mathematics. By J. F. Scott. Pp. 266. 63s. 1958. (Taylor and Francis)

This is a general survey of the history of mathematics, covering ground similar to that of Struik's *A Concise History of Mathematics*, but in much greater detail. The long account of seventeenth century mathematics, on which Dr. Scott is expert, is valuable, but elsewhere the book must be used with considerable caution. In particular, the bibliography is useless as a guide to the present state of the literature on the history of mathematics.

M. A. Hoskin

Geschichte der Mathematik. By J. E. Hofmann. I : Von den Anfängen bis zum Auftreten von Fermat und Descartes, pp. 200, (Sammlung Göschen Band 226, 1953) ; II : Von Fermat und Descartes bis zur Erfindung des Calculus und bis zum Ausbau der neuen Methoden, pp. 109 (Band 875, 1957) ; III : Von den Auseinandersetzungen um den Calculus bis zur Französischen Revolution, pp. 107 (Band 882, 1957). (Berlin : Walter de Gruyter)

This set of three small paper-backed booklets, costing only a few shillings in all, provides a highly concentrated survey of the history of mathematics. Although general issues are not ignored, the great merit of this work lies in the

extraordinary wealth of detailed knowledge which the author displays. This is particularly the case with the magnificent bibliographies, which make the work indispensible to the serious student of the subject.

An English translation of Volume I, but without the bibliography, has been published in New York by the Philosophical Library at $4.75.

M. A. Hoskin

The Gentle Art of Mathematics. By D. Pedoe. Pp. 143. 15s. 1958. (English Universities Press)

This is a delightful book which makes a valuable contribution to the important task of giving the general reader a glimpse of the mathematicians' collection of jewels. Of special interest are two letters from Isaac Newton to Samuel Pepys discussing the relative chances of throwing a six with six dice or two two sixes with twelve dice.

Apart from puzzles like the wonderful counterfeit coin problem there are beautifully well written sections on probability, topology and Boolean Algebra. It is perhaps worth remarking that in the section on transfinite numbers the inequality defined is weak, not strong inequality, since every infinite set may be one-one correlated with a proper part of itself. On the relation of logic to mathematics the author reminds us that a training in mathematics is no protection against illogical thinking, and proves his point on the same page in a discussion of the Liar paradox.

The book may be warmly recommended to a wide circle of readers.

R. L. G.

Determinanten und Matrizen. By R. Kochendörffer. Pp. 144. DM 6,60. 1957. (Teubner, Leipzig)

This admirable little book presents an elegant and interesting introduction to linear algebra ; a great deal of information is given without losing sight of the needs of the first year student for a leisurely treatment. By comparison with an English text book there is however a shortage of examples.

R. L. Goodstein

Analytic geometry problems. By C. O. Oakley. Pp. xviii, 253. $1.95. 1958. (Barnes and Noble, New York)

Teachers often find it useful to have sets of problems to supplement those given in textbooks. This book gives nearly 700 problems in analytical geometry, of which about half are worked, and each chapter has a résumé of the theory. The book covers the whole of the usual British grammar school course, and goes a little way beyond : for example, there are a few chapters on three-dimensional geometry.

I noticed only two serious errors. Firstly, for a general conic

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0,$$

with eccentricity e, the identity $B^2-4AC=4(e^2-1)$ is given (twice). This is obviously incorrect : all that is needed is the true result that e^2-1 has the same sign as B^2-4AC. Secondly, the author makes the surprising statement that the locus of the point of intersection of perpendicular tangents to the hyperbola $x^2/a^2-y^2/b^2=1$ is only *part* of the circle $x^2+y^2=a^2-b^2$, and the diagram implies that the parts of the circle in two of the four regions of the plane determined by the asymptotes are to be omitted. What actually happens is that from points on these parts of the circle the tangents touch the same branch, whereas from points on the other parts the tangents touch opposite branches.

E. J. F. Primrose

Elements of Modern Abstract Algebra. By K. S. Miller. Pp. 188. 40s. 1958. (Hamish Hamilton)

Grundzüge der Algebra. Part I. Allgemeine Gruppentheorie. By H. Lugowski and H. J. Weinert. Pp. 234. DM 10,00. 1957. (Teubner Leipzig)

These are both quite good introductions. Miller's book covers ring and field theory as well as group theory, whereas Lugowski and Weinert give a separate volume to group theory. The German book is a quite outstanding piece of exposition which one hopes will soon find a translator. The pace of Miller's book is necessarily faster and there is less time to bind concepts together. For instance for lack of reference to the transformation of the elements of a group by an element the term " invariant subgroup " loses its meaning. Both books rightly do not rest content with trivialities but prove results of some depth, and both may be warmly recommended to first-year undergraduates. Examples are plentiful and the German text contains 27 pages of solutions.

R. L. Goodstein

Statistics for the Social Sciences. By T. G. Connolly and W. Sluckin. Pp. viii, 166 : 16s. 1957. (Cleaver Hume Press Ltd.)

The first edition of this book was reviewed in the *Mathematical Gazette* for February 1954. The second edition includes a chapter on the analysis of variance ; but apart from this there are only minor alterations to the text. Despite its title, this book is essentially an elementary introduction to statistical method which could have been written for any group of students whose knowledge of mathematics is slight.

Freda Conway

Jubilee of Relativity Theory. Ed. A Mercier and M. Kervaire. (Basel 1956)

Einstein spent the years 1895–1910 and 1912–1913 in Switzerland, became a Swiss citizen in 1901, did much of his most important work there and, in particular, he wrote his famous first paper on relativity theory in 1905 when he was employed in the Swiss patent office in Berne. Therefore it was natural that the jubilee of relativity should be celebrated at a conference organized by Swiss scientists and that the city of Berne should be chosen as its meeting-place. It was also natural, but possible only through the generosity of the Swiss authorities, that the participants should be drawn from all over the world. The outcome was, in fact, the first international conference ever to be devoted to relativity theory (thus affording a semblance of truth to the tourist who, after enquring about this unusual gathering, was heard to report to his friends, " Oh, it's some society that meets once every fifty years "). The present volume is the report of the proceedings and contains the texts of almost all the nine longer lectures and twenty-five shorter communications together with the ensuing discussions, the opening and closing presidential addresses by W. Pauli, and the addresses by L. Kollros and Pauli at a public celebration following the conference. The occasion proved a great stimulus to those who were privileged to attend ; the publication of the proceedings will now convey much of this stimulus to a much wider circle, thanks to the devoted editorial labours of A. Mercier and M. Kervaire.

Many times during the fifty years concerned it began to look as though nothing of much significance was left to be done in relativity theory as such, but always some unexpected development revitalized the subject. The present publication will show that it is now more alive than ever and, indeed, that the significance of some of its most fundamental problems is now only beginning to be realized, the solutions of these problems being still awaited.

The subjects dealt with can be classified roughly as direct applications, fundamental problems and extensions of the theory. Brief mention of some of the salient points may be made here, but obviously a review paper-by-paper is not possible. Even though some of the contributions are quite short, it ought to be stated that most of them include valuable bibliographies of the topics concerned.

Direct applications include the three classical " tests " of general relativity. The observational results on the two concerning radiation, the bending of light-rays in a gravitational field and the gravitational red-shift, were reviewed by R. J. Trumpler. On the whole, the evidence appears not to contradict the theoretical predictions, but doubts were voiced by E. F. Freundlich. The trouble about these tests is that they depend in practice upon minute effects that are difficult to isolate. Trouble of this sort ought not to arise in the application of relativity theory to cosmology, where by far its most important direct consequences are to be expected. But that subject has its own formidable observational difficulties ; these were reviewed by W. Baade, though his address is one of the few not published. H. P. Robertson discussed the current state of relativistic cosmology. It is probably true to say that the relativistic treatment of space-time is essential and adequate for the large-scale study of the universe, but that theoretical predictions depending also upon the field-relations cannot yet be accurately tested.

Amongst the most fundamental work in relativity theory in recent years has been that done by A. Lichnerowicz and his school on boundary-value problems for the field-equations. A general account was given by Lichnerowicz. He also outlined some of the " global " problems of relativity about whose solution little is known at present except in special cases. In the reviewer's opinion such problems are the most vital ones in relativity. For we cannot be satisfied with any relativisitic treatment of a particular physical problem unless we know that it can be correctly fitted to an acceptable universe, and it is doubtful whether we can yet say even what we should regard as acceptable for this purpose. Other urgent problems that are now receiving attention but which are still only partially understood are those of the origin of inertia and the existence and nature of gravitational waves, problems here represented by the contributions of F. A. E. Pirani and N. Rosen, respectively.

In a general sense, almost all attempts to extend relativity theory aim at some kind of reconciliation with quantum theory. Much effort has been expended in attempts at the more or less direct quantization of general relativity as described by P. G. Bergmann. However, it is uncertain whether such work can be well-directed until the problem of gravitational waves in non-quantized relativity has been solved. O. Klein described more indirect approaches from a five-dimensional relativity theory but these seem so far only to have accentuated the difficulties. A profound comparative analysis of the foundations of quantum theory and relativity theory was given by E. Wigner. In particular, he gave a penetrating analysis of the rôles of coordinates in the two theories, and offered some very tentative suggestions for the formulation of a quantum theory without the use of coordinates. As is well-known, Einstein himself rejected the basic concepts of modern quantum theory and believed that a unified field-theory must exist from which much the same results could be got in a classically deterministic manner. The mathematical structure of Einstein's final efforts to develop such a theory was described by B. Kaufman. The theory has a certain mathematical interest, although few physicists think that progress in understanding or describing the physical world is to be achieved by means of it.

Most of the other papers dealt with significant mathematical or physical questions coming under one or other of these three heads.

This brief survey will inevitably give the impression of inconclusiveness in most of the contributions. But this is only natural since they were concerned with present problems rather than with past achievements. Relativity theory has, of course, effected a permanent transformation in the whole of fundamental physics. The way in which it has done so might have been the theme of the proposed lecture on Physics and Relativity by Neils Bohr. However, he was unable to attend and in his place Max Born gave a most charming and vivid account of his contacts with Einstein and of the scientific world into which Einstein's ideas were introduced. Born's contribution to the volume is a document of much historical value. So also is the account of Einstein's early scientific life by his fellow-student L. Kollros in the address already mentioned; it shows how much the world owes to the friendliness, tolerance and encouragement that Switzerland gave to the unknown and unorthodox young man from another country. Finally, this remarkable volume will be valued still further for the several acute commentaries by Pauli.

W. H. McCrea

Vorlesungen über Himmelsmechanik. By C. L. Siegel. Pp. 212. DM 29. 80. 1956. (Springer, Berlin)

This is a book by a very distinguished mathematician on the mathematics of some general fundamental problems of celestial mechanics, not a manual for the working astronomer.

The basic mathematics employed is largely the theory of (real and complex) series, together with a little point-set theory, mostly in the third section. On the whole it is not as heavy as might be expected, thanks to the author's skilful manipulation. He tells us that the work depends upon results obtained during the past 70 years concerning the behaviour of solutions of differential equations in the large. This is borne out by comparing Siegel's review of the subject with one, not mentioned in this book, made nearly 60 years ago by E. T. Whittaker (" Report on the progress of the solution of the problem of three bodies " *British Association Report* 1899, pp. 121–159). It can be seen that the root-ideas of most of the later developments had been discovered at that time, but these developments have depended upon subsequent more powerful mathematical techniques, to which Professor Siegel himself has been a notable contributor.

W. H. McCrea

La Dynamique Relativiste et ses Applications. By H. Arzeliès. (Paris: Gauthier Villars 1957)

The first volume of this series was reviewed in *Math. Gaz.* **41**, **304**, **1957**. The present volume is one of four that the author promises on relativistic dynamics and even in all of these he apparently intends to confine himself to the special theory of relativity. In this volume he discusses only the motion of a particle having small acceleration, in the case of an arbitrary force and in the case of a charged particle in an electromagnetic field.

Everything said in the earlier review about the general character of the first volume applies equally to this one. In particular, the copious historical notes again form a novel and useful feature.

W. H. McCrea

A compendium of mathematics and physics. By DOROTHY S. MEYLER and SIR GRAHAM SUTTON. Pp. x, 384. 25s. 1958. (English Universities Press)

This new volume in the series of Physical Science Texts edited by Sir Graham Sutton, is a reference work covering a wide field of topics, ranging from about A level to a General Honours Degree standard, two-thirds pure mathematics, one-third applied mathematics and physics. Miss Meyler's pure mathematics starts with elementary algebra and goes far enough to include the Mainardi-Codazzi equations, but it should be noted that there is nothing about special functions such as elliptic functions or the functions of Legendre and Bessel. Possibly the inclusion of such material would have swollen the book unduly, since its plan makes provision for a good proportion of connective and explanatory tissue as well as a collection of formulae. This comment is carefully woven into the structure, and such slips as there are, for example, the remark concerning $(\sin z)/z$ on p. 176 and the comment on ∇ on p. 242, seem to be the result of a natural striving for compression. Sir Graham Sutton's compact notes on physics should not be criticised by one who is no physicist; it will be enough to note that Sir Graham has found room for four pages of meteorology.

T. A. A. B.

Generators and relations for discrete groups. By H. S. M. COXETER and W. O. J. MOSER. Pp. viii, 155. Springer-Verlag 1957. (Ergebnisse der Mathematik, Neue Folge, Heft 14)

The title of this book describes its content very aptly. It is a veritable treasury of information. Everything that is known to-day about "abstract" definitions of groups, that is, definitions in terms of generators and relations connecting them, is deposited here. A wide variety of methods, from algebra proper, geometry both euclidean and non-euclidean, topology, crystallography etc. have been used in these researches and are lucidly described here. There are many diagrams, excellent tables, and the book is beautifully produced.

K. HIRSCH

The Theory of groups. By HANS ZASSENHAUS. Second edition. Pp. x, 265. New York, Chelsea Publishing Company 1958

The second edition of this well known book differs from the first by the addition of new material both in the text and in some 70 pages of appendices. The main new topics are semigroups, the theory of lattices and its applications to group theory, the construction of groups by generators and relations. Also many new exercises of varying degrees of difficulty have been added. All in all the book is now again much more up-to-date; it is a pity that the author's terminology in the theory of homomorphic mappings has not been brought into line with present usage.

K. HIRSCH

Sur les bases des groupes d'ordre fini. SOPHIE PICCARD. Pp. xxiv, 242. Neuchatel, Secretariat de l'Université, 1957.

If a finite group can be generated by k, but not by fewer than k, of its elements, it is said to have a basis of order k. The author has devoted her life work to the study of the question how many different bases a given group possesses, more generally, in how many distinct ways it may be generated by a fixed number of its elements such that none of them is superfluous. The present book gives a complete account of her findings. The topic is of rather specialist interest. But it may surprise some readers to learn that even such a harmless group as the symmetric group S_4 of the 24 permutations of four objects can be generated in 108 distinct ways by two of its elements and in 148 distinct

ways by three of its elements. The symmetric and alternating groups of degrees up to 10 form the major part of the investigations, and the complexity of the situation increases rapidly with the degree. K. HIRSCH

Statistics: A New Approach. By W. ALLEN WALLIS and HENRY V. ROBERTS. Pp. xxxviii, 646. 50s. 1957. (Methuen)

This book, first published in the United States in 1956, has now been issued in this country by a British publisher. In it the authors are grappling with a problem familiar to most teachers of statistics, the problem of teaching the subject to students with no knowledge of (and often an aversion for) mathematics. The novelty of the approach consists not so much in eliminating mathematical proofs, which many teachers have, often with regret, been forced to do, but in trying to make a virtue out of this necessity and arguing that the approach is universally suitable. The authors feel that, even for those who are specializing in mathematics, " the great ideas of statistics are lost in a sea of algebra ". They also assert that such students " seldom know their mathematics well enough to use it as a medium for learning another subject ". Since the teaching of statistics to the level of this book (approximately that of a one year Subsidiary Course at a University) can be made to involve only Algebra and Calculus to Advanced G.C.E. level, such statements, if true, must be an indictment of the teaching of mathematics in the United States. To teach statistics by the methods used here (that is by dogma illustrated by examples) to students, who should have the necessary mathematical equipment to deal with a more sophisticated approach, would seem to the reviewer to deny such students a chance of furthering their mathematical education and of practising mathematical techniques, which all too often become rusty with disuse.

The book itself is divided into four sections under the headings: The Nature of Statistics, Statistical Description, Statistical Inference and Special Topics. The first part includes two long chapters on the uses and misuses of statistics, a chapter on sampling ideas and finally one on observation and measurement. The second is concerned largely with the summarizing and presentation of data and with variation and association. These two sections (a total of 306 pages) are, to a statistician, largely common sense, but they could, and perhaps should, be read by any intelligent person wishing to understand the complications of civilization. Students of economics and the social sciences, at whom this book is primarily aimed, would undoubtedly be the better for reading them.

The third section contains the meat of the book, introducing probability, sampling distributions, significance testing and estimation. In discussing the various test procedures, the authors go to astonishing lengths to reduce the familiar t-, R- and χ^2- tests to tests based on a normal distribution. This has the advantage that it avoids the necessity of including tables of these distributions but, since the reduction of the χ^2- test to normal form involves the calculation of a cube root (although a table of square roots is given, no table of cube roots is included), this advantage seems questionable. It is hard to believe that, for example, the χ^2 distribution is more difficult to understand than the normal distribution, when no mathematical derivations are involved.

The final section is a collection of special applications including the design of experiments, quality control, regression (both linear and curvilinear), time series and short cut methods (including ranking methods). Even more than the previous sections, this is a series of recipes justified, if at all, by a verbal discussion.

Summarizing, the book has, for the non-mathematician, much to commend

it. The phenomenal number of well-chosen examples (both to illustrate the text and for the reader to do for himself) is a necessary adjunct to this method of teaching statistics, and teachers of the subject will be grateful for them. The mathematician will probably gain little from this book, except, perhaps, an increased respect for the brevity and power of mathematics.

F. DOWNTON

Analytische Geometrie. By K. P. GROTEMEYER. Pp. 202. DM. 4.80. 1958. (Walter de Gruyter, Berlin)

Sammlung Göschen is a series of small paper-backed books on various subjects in arts, science and technology. They are very well printed on good-quality paper, and most of them cost the equivalent of just over four shillings (the book under review counts as a double volume).

Analytische Geometrie is a first-class account of three-dimensional geometry, treated by the methods of vector and matrix algebra (no previous knowledge of vectors or matrices is assumed). For most of the book the geometry is euclidean, but the last two of the twelve chapters deal with projective geometry. Everything is clearly explained, and some excellent diagrams help to illustrate the theory.

E. J. F. PRIMROSE

Introduction to abstract algebra. By C. RACINE. Pp. x, 194. 6 rupees. 1957. (S. Viswanathan, Madras 31)

This book gives an account of the theory of groups, rings, fields, and matrix algebra. The author says in the preface that it contains very little new material : it is intended to be a simple and clear presentation of existing knowledge.

The book is a masterly survey of the subject, but it seems unsuitable for beginners, for whom it is intended. The author has attempted to do far too much in a book of this size : some of the material is beyond the scope of van der Waerden's two volumes. The result is that he has to go at a tremendous pace. More examples, especially simple ones, would have helped a beginner to appreciate the abstract theory.

In a book which contains so many definitions, an index would have been helpful.

E. J. F. PRIMROSE

Differentialgeometrie. By ERWIN KREYZIG. Pp. xi, 421. 1957. (Akademische Verlag, Leipzig)

This introductory textbook deals with properties of curves and surfaces imbedded in 3-dimensional Euclidean space. After an introduction chapter on vector calculus, the author deals carefully with the elementary properties of skew curves. Chapter III is headed " Surface Theory : The first fundamental form : Foundations of tensor calculus ". Chapter IV deals with properties involving the second fundamental form, Chapter V with geodesic curvature, Chapter VI with special transformations *e.g.* conformal transformations, Chapter VII with absolute differentiation, Chapter VIII with special surfaces.

The reviewer's main criticism is with those parts of the book which deal with tensors. In spite of the title of Chapter III, tensor calculus proper is not considered until Chapter VIII. The definition of a tensor (pp. 116–119) as a set of quantities which obey a prescribed transformation law when the coordinates are changed is not entirely satisfactory. A more important criticism is that tensors are defined only at points in a Riemannian space. A student may get the false impression that it is necessary to work in a Riemannian space before one can use tensors, whereas in fact the Riemannian structure

itself arises by associating with each point of a differentiable manifold (in this case an n-cell) a non-singular 2nd order covariant tensor. An improvement would have been obtained by first considering tensor algebra *i.e.* the properties of tensors at a point, and then dealing with fields of tensors and finally tensor calculus.

The book is well printed, well illustrated, and is easy to read. The 40 pages devoted to worked solutions of examples will be found very helpful to many students. The style of the book, however, reflects the view point of differential geometry of the early 1920's. T. J. WILLMORE

Analytische und konstruktive Differentialgeometrie. By ERWIN KRUPPA. Pp. vii, 191. 1957. Springer-Verlag. Vienna.

The first half of this book is a self-contained account using vector methods of the classical local differential geometric properties of curves and surfaces imbedded in 3-dimensional Euclidean space. This theory is applied in the second half to problems of constructive differential geometry involving curves, ruled surfaces, special surfaces and kinematic differential geometry. The exposition is concise but clear, and the description is further clarified by 75 carefully drawn diagrams. The printing and general layout of the book are excellent. T. J. WILLMORE

A Calculation of Atomic Structures. By D. R. HARTREE. Pp. xiii, 181. 40s. 1957. (Wiley, New York)

The main purpose of this book is to describe in considerable detail the methods developed by the author for getting numerical solutions to the integro-differential equations for atomic radial wave functions, making the assumption that each electron is in a stationary stage in the field of the nucleus and the other electrons. The general strategy of the process has become well known under the name of " method of self-consistent field ". However, to make the work reasonably self-contained Professor Hartree has provided several most informative general chapters : one is on atomic structures, Schrödinger equation and wave mechanics ; another on variational methods ; and a third on numerical techniques in which, starting from the definition of finite differences, methods for interpolation, quadrature and integration of differential equations are developed. The book is very readable and as is to be expected from an author who has actually done all the computing himself—much of it on a small hand machine, in fact—has a very practical air ; there are many useful tips for the practical mathematician scattered through the work. Altogether, in addition to providing the only complete account of an important piece of specialised technique, the book provides very rewarding reading for anyone interested either in general ideas on atomic structure or on numerical methods. J. HOWLETT

Games and Decisions. By R. DUNCAN LUCE and HOWARD RAIFFA. Pp. xi, 509. 70s. 1957. (J. Wiley & Sons)

From a thorough study of the book—which it deserves—one gains the impression, certainly intended by the authors, that the applicability of the theory is, as yet, open to doubt. However, just as physical problems have contributed to the development of mathematical theory, and as the latter has dealt with increasingly complex situations, a more elaborate mathematical development centred on social sciences has started with game theory, and mutual benefit may confidently be expected in this field as well.

Many new topics in game theory have been developed since the publication of von Neumann's and Morgenstern's classic tract—where, incidentally, the mathematics are more elementary though more rigorous than in this book—and nearly all of them are dealt with here. We mention, in particular, " Individual Decision Making Under Uncertainty " and " Group Decision Making " (the titles of the last two chapters), because these have the characteristic flavour of the new " social science mathematics " which even pure mathematicians should not ignore. Also, the distinction between (von Neumann's) cooperative and (Nash's) non-cooperative games is stressed and studied in various relevant social contexts.

Without any doubt, " Games and Decisions " will remain, for many years to come, a useful textbook in its subject. If its references become soon out-of-date, then this will in a large measure be due to the stimulus to further development which the book itself provides. S. VAJDA

Geometric Integration Theory. By HASSLER WHITNEY. Pp. xv. 317. 68s. 1957. (Princeton University Press and Oxford U.P)

This highly important book treats multiple integrals from a geometrical standpoint : coordinate systems are secondary, the primary interest is in the relationship between the integral and the domain of integration. The main tools are those of classical analysis with the theory of differential forms and cohomology ; these are expounded *ab initio* so that the book could be read by any Honours graduate and is a good introduction to topology.

The first part develops the theory of differential forms and their integrals, leading up the general Stokes' theorem and de Rham's theorem. Analysts will find much of interest here : *inter alia*, fresh and meticulous proofs of theorems on implicit functions and change of variable in integration. The second and third part deal with researches of the author and his school, in which the integral is characterised as a linear functional on chains of polyhedra having certain continuity properties, and the corresponding integrands are sought. Throughout, great pains have been taken to ensure accuracy and readability. J. L. B. COOPER

Grundlagen der Mathematik in Geschichtlicher Entwicklung. by OSKAR BECKER. Pp. xi, 422. (Karl Alber, Freiburg/München)

The conception of this book is wholly admirable. The number of books on the history of mathematics is growing steadily, and the journalistic re-writing of general accounts of the development of the subject may well help to spread the interest in some of the problems of the history of ideas. But here we have a historical collection of original papers that is relevant to mathematics as such : what did the discoverers themselves, or other great mathematicians think about the subject, and what, then, are now considered to be the proper foundations. History thus fulfills a truly mathematical function, namely to analyse the structure and to secure the fundaments. The documents of this source-book start with examples of Egyptian and Babylonian mathematics to show how it all began ; with the Greek contribution we get both the first-fruits of mathematical method proper and the philosophers' thoughts about them. A brief chapter on the beginnings of the calculus leads on to the main part on the 19th and 20th centuries : Riemann, Pasch, Poincaré, Hilbert on geometry ; Dedekind and Cantor on number and sets ; finally Russell, Brouwer, Gentzen and Lorenzen on logic and the theory of proof. It is not an easy book, and the editor's connecting notes, though excellent, presume a mature reader who will see where it all leads, where an attempt to lay good foundations has failed, what part of a theory is still valid after the need for

greater rigour has become apparent. Probably the book should be used in a seminar where discussion will fill the gaps. It is, of course, in German, with a justifiable bias for original German texts, and while all other authors are translated into German, the accurate references (*e.g.* for Aristotole and the other Greeks) will enable the English reader to use a translation of his own choice. Where so much is given (and much of it unfamiliar) it would be ungenerous to quarrel with the selection ; we might all find a favourite author of whose contribution we think very highly ; Barrow, for instance, seems to deserve a larger share, and there is perhaps a bit too much of Georg Cantor—yet how exciting are just these passages, both in themselves to us now, and in their impact on Cantor and his contemporaries who were well aware of their ancestry. One seems to hear the very same phrases which the Greeks used, and the struggle is still for the clarification of the same concepts, number, space, continuum. By letting each author speak in his own words, Becker manages to revive this historical awareness of continuity ; but his greatest merit I believe is the attitude in which he makes us look at the history of our subject : not how much further we have gone than earlier generations, but how great were the men who came before us and whose inheritance is ours.

A. PRAG

The Elements of the Theory of Real Functions. By J. E. LITTLEWOOD. Pp. 71. 9s. 6d. 1956. (Heffers, Cambridge. Dover, New York)

Hypothèse du Continu. By W. SIERPINSKI. Pp. 274. $ 4.95. 1957. (Chelsea, New York)

In the preface to the second edition of these famous notes on the elementary theory of sets, Littlewood commended the practice of providing lecture notes in advance. Anyone who had the great good fortune to attend this course of lectures will remember how the printed notes served, not as a summary, but as a subject for criticism and comparison. For the first time the student saw mathematics not as something from the past, finished and perfected, but as a subject which grew before his eyes ; he learned the important lesson that print did not give mathematics its authority but that each created it anew for himself.

Sierpinski's important book is a reprint of the 1934 Edition with a very valuable 90 page appendix reproducing 16 papers published by Sierpinski during the past 20 years, chiefly in *Fundamenta Mathematicae.*

R.L.G.

Variational Methods for Eigenvalue Problems. By S. H. GOULD. Pp. xiv, 179. 48s. 1957. (University of Toronto Press and Oxford University Press)

The best description of this book is given by the author himself when, on p. 30, he writes : " Calculation of the eigenvalues of a complicated physical system is an extremely difficult problem. The purpose of this book is to discuss certain theories which have been developed to that end ".

The author commences by discussing the oscillations of a dynamical system and shows how the normal modes and frequencies can be naturally generalized into the wider concepts of eigenfunctions and eigenvalues (or eigenfrequencies). The ideas of metric spaces and linear operators are then introduced and explained in a very readable and simple manner. Eigenfunctions and their corresponding eigenvalues are then defined, as usual, by means of the equation $Hu = \lambda u$, where H is a linear operator. The author then proceeds with the exposition of the Rayleigh-Ritz Principle.

Rayleigh's famous theorem states that if S denotes a physical system and S_1 denotes the same system with some constraint applied to it then the eigen-

values (or eigenfrequencies) of S_1 are higher than those of S. For example, an uncracked glass has a higher ring than a cracked one, since the uncracked glass is constrained relative to the cracked one. In general, if a system has a finite number of eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ and on applying i arbitrary constraints we obtain a new system with eigenvalues $\lambda_1' \leqslant \lambda_2' \leqslant \ldots \leqslant \lambda'_{n-i}$ then for every $m \leqslant n-i$ we have $\lambda_m \leqslant \lambda'_m \leqslant \lambda_{m+i}$; a result which can be extended to systems with an infinite number of degrees of freedom. Thus if the eigenvalues of a system are too difficult to find, constraints are added until the eigenvalues of the new system can be computed, and these will form upper bounds for the eigenvalues of the original system. After proving these results the author gives many applications to problems of vibrating rods, membranes and plates. He uses many of the concepts of Hilbert space theory to shorten and simplify the discussions, and, since these concepts are carefully explained, the reader can learn both eigenvalue theory and Hilbert space theory together by easy stages.

All this material occupies the first five chapters of the book. Except for a much too condensed account of Lebesgue integration it is all lucid and carefully arranged so as to make easy and attractive reading.

The remainder of the book is largely devoted to the Weinstein method of dealing with the eigenvalue problem. In the Weinstein method constraints are removed and lower bounds are obtained for the eigenvalues.

To undertake the exposition of such a large subject as eigenvalue theory in a book of only 179 pages is no light task, but there is no doubt that this work is a welcome addition to the literature of the subject.

Charles Fox

Algebraic Geometry and Topology, a symposium in honor of S. Lefschetz. Edited by R. H. Fox, D. C. Spencer and A. W. Tucker. Pp. viii, 399. 60s. 1957. (Princeton University Press)

This volume was produced to celebrate the seventieth birthday of S. Lefschetz. It consists of survey articles by W. V. D. Hodge and N. E. Steenrod, describing Lefschetz's contributions to algebraic geometry and to topology, a bibliography of Lefschetz's works, and some two dozen papers by various authors, on topics which have developed out of the fundamental work which Lefschetz initiated.

The reputation of a mathematician rests in the long run on his influence on his successors, and on the extent to which his ideas lead to further developments of his subject. Judged by these criteria, Lefschetz occupies a leading position among his contemporaries. The subject of algebraic topology, which has in recent years undergone such striking developments, may be fairly said to be his creation, for it was Lefschetz himself who, in the twenties and thirties, first attempted a systematic exposition of a general theory. Many of the dominant ideas in modern algebraic geometry, notably the theory of harmonic integrals, can be traced back to his Borel tract of 1924, which made the first serious study of the topological properties of algebraic varieties. A careful study of the surveys of Lefschetz's work in this volume will show how many of the tools which he used to develop a general topological theory had their origins in the algebraic geometry of the Italians, but the debt which topology thus owes to algebraic geometry has been repaid in full measure, not least by Lefschetz's own great contributions to the transcendental theory of varieties.

This volume is a notable tribute to one of the pioneers in two fields which are today among the major branches of mathematical research.

J. A. Todd

Integral equations and their applications to certain problems in mechanics, mathematical physics and technology. By S. G. Mikhlin. Translated from the Russian by A. H. Armstrong. Pp. xii, 338. 80s. 1957. (Pergamon Press)

This book is a translation of the second Russian edition, which appeared in 1949. The first of the two parts is devoted to general theory, and the second to applications.

The exposition of the Fredholm theory, of the Hilbert-Schmidt theory for symmetric kernels, and of the theory of singular equations, *i.e.* those involving a Cauchy principal value integral, is fairly standard. It is assumed in the Fredholm theory that the integral of the square of the kernel with respect to one variable is a bounded function of the other; this is sufficient to assure uniform convergence of most of the series involved. Kernels with weak singularities, possessing an iterate satisfying the above condition, are also considered. The author gives his own interesting proof of the representation of the Fredholm resolvent as the quotient of two integral functions. Particular attention is paid to methods for obtaining numerical results.

The applications discussed are mainly to problems arising in fluid dynamics and elasticity theory. The Fredholm theory is applied to the solution of elliptic partial differential equations in simply- and multiply-connected domains in two and three dimensions. Many special problems are discussed in considerable detail. The Hilbert-Schmidt theory is applied to vibration and other critical value problems of familiar types. The applications of the singular theory are similar to those given in N. I. Muskhelishvili's book (*Singular integral equations*, Amsterdam, 1953).

The book will be very useful to anyone who wishes to see how integral equations can be used as a practical tool in applied mathematics.

A number of references appear in the text with no corresponding item in the bibliography at the end; it is to be hoped that this oversight will be corrected when a reprint becomes necessary.

F. Smithies

Engineering Analysis. By S. H. Crandall. Pp. x, 417. 71s. 6d. 1956. (McGraw-Hill)

The subject matter of this book is applied mathematics, where the adjective means what it says and is no mere conventional description. The plan of the work recognises that the essential divisions of this subject are in the mathematics rather than in the applications, and that identical mathematical techniques cut across the more customary subdivisions of the field of engineering.

A vast ground is covered as a result of obviously severe and successful self-discipline in the writing. Many general theorems which bear upon computation procedures are enunciated only, in precise terms quite sufficient for purposes in view. Abundant references indicate sources for the omitted proofs, as well as for further developments of many techniques themselves here treated in considerable detail. There are illustrative examples fully worked in all sections, as well as many for the reader, mostly with answers supplied. Explanations are particularly clear; the technique and the objectives of each stage in the computation methods are explained together, and solutions are discussed to develop physical insight together with mathematical techniques.

Mathematical knowledge corresponding to an engineering degree or a first mathematical degree is required of the reader, but the book includes *ab initio* accounts of the parts of matrix algebra, of the calculus of variations and of the method of characteristics which are sufficient to develop the techniques of chapters 2, 4 and 6.

The book is warmly recommended alike to students and to established engineering technologists, especially for unassisted study ; the reader receives an impression of individual tuition from an author who is expert not only in his subject, but—no less importantly—in its exposition as well. The American engineering societies are to be congratulated on sponsoring this volume in their excellent series. THOS. H. O'BEIRNE

Angular Momentum in Quantum Mechanics. By A. R. EDMONDS. Pp. 146. 30s. 1957. (Princeton University Press)

This is an advanced text aimed at the specialist theoretical reader who is already familiar with a more conventional textbook of quantum mechanics. The first three chapters discuss some group theoretical preliminaries, the definition of angular momentum in quantum mechanics and its matrices and the important central problem of the coupling of angular momentum vectors. This last topic is treated for j integral and half-integral, and the vector coupling coefficients are defined in terms of the recent $3j$-symbols of Wigner which are used from then on. The remaining four chapters apply the algebra to various problems involving rotational properties. There is a chapter on the coupling of three and four angular momenta.

This is a successful unified treatment of an important field. The arguments seem in places unnecessarily condensed, but the book's strength lies in the applications in the later part which includes a careful discussion of various notations in the literature and offers a universal symmetric symbolism.

H. C. BOLTON

Les principes de la statistique mathématiques. By R. RISSER and C. E. TRAYNARD. Pp. xvi, 195. 3500 Fr. 1957. (Gauthier-Villars Paris)

After assembling the basic tools, the authors discuss Pearson curves, estimation, analysis of variance and sampling methods. Numerical examples are given in the earlier sections but modern techniques are left at the theoretical stage. The book is based largely on the work of the English school, and similar but better accounts are available to readers in this country.

R. L. PLACKETT

Les principes de la statistique mathématiques. By R. RISSER and C. E. TRAYNARD

Livre II : Corrélations. Séries chronologiques. Pp. xi, 418. 1958

In their second volume, the authors explore most of the ramifications of correlation from its beginnings to the present day, and their subjects range from simple indices to the estimation of confidence bands for the spectral function of a stationary process. They make no attempt at selection and the result is a discursive historical review rather than an organised modern text-book. R. L. PLACKETT

An Introduction to Algebraic Topology. By A. H. Wallace. 40s. (Pergamon Press)

This book has already been widely welcomed and will prove extremely useful to students who want to learn Homology Theory. The knowledge presupposed is modest and yet in under 200 pages the Singular Theory is systematically developed. The first three chapters introduce those notions of Analytic Topology necessary for Algebraic Topology. There is a chapter on the Fundamental Group and this is followed by the four chapters on Singular Homology Theory that constitute the heart of the book : in these chapters all the properties taken by Eilenberg and Steenrod as axioms are verified. In a

final chapter simplicial complexes are defined and it is shown how the combinational structure can be used to calculate the singular homology groups.

A particular and praiseworthy feature of the book is the care taken to motivate the concepts and the proofs : the discussions that precede the proofs are honest and penetrating. The author indeed remarks in his preface that he does not assume much mathematical maturity in his readers and that topology is an ideal field in which to develop this maturity.

The chapters on Analytic Topology are the least satisfactory in the book. There is a systematic infelicity in handling neighbourhoods and open sets. There is no explicit discussion of bases for open sets, but internal evidence suggests that they were prominent in an earlier draft or that a different definition of neighbourhoods was used. The definition on page 17, for instance, says that a point p of A is *interior* to A if there is a neighbourhood U of p such that $U \subset A$; but this is equivalent to the simpler statement that A is itself a neighbourhood of p. This point of view persists and needlessly complicates several proofs. Again it is surprising to find in this condensed discussion of Analytic Topology anything about limit points of sets ; they have surely very little relevance to Algebraic Topology and the closure of a set can be defined directly (and better) without appeal to this notion.

The chapter on the Fundamental Group is peppered with splendid diagrams. Indeed throughout the book everything to do with the geometry of the subject is excellently handled. In this chapter however the author should have left more of the details to the reader or should have given some thought to simplifying them. He writes down homotopies explicitly and the nadir is reached on page 87 where, for a single homotopy, five lines of description are needed one of which is

$$F(S, t)=f(qs/(8t+1)-(4-4t)/8t+1), \ \tfrac{4}{8}(1-t) \leqslant s \leqslant \tfrac{1}{8}(5+4t).$$

(There is incidentally no proof of the theorem in Analytic Topology that justifies the piecing together of consistent maps defined on closed subsets). Since simplicial complexes are suppressed until the last chapter, virtually nothing can be done in this one about calculation : it is, for instance, still an open question at the end of this chapter whether the fundamental group of a circle (known to be cyclic) is infinite or finite.

The chapters on homology theory are very good indeed. A reader with some idea of Analytic Topology and no curiosity about the Fundamental Group can start at Chapter 5 without embarrassment. He will find a clear and thorough description of the Singular (Simplicial) Theory and its principal properties.

S. Wylie

Einfuhrung in die transzendenten Zahlen. Von Th. Schneider. Die Grundlehren der mathematischen Wissenschaften, Band 81. Pp. 150. DM 24.80. 1957. (Springer, Berlin)

Transcendental numbers are (real or complex) numbers that are not algebraic, *i.e.* that do not satisfy an algebraic equation with rational coefficients. Their existence, conjectured already by Euler, was first proved by Liouville in 1844. The first interesting examples were, however, due to Hermite and to Lindemann who proved the transcendency of e in 1873 and of π in 1882, respectively. The next great progress came with Siegel's proof of the transcendency of the Bessel functions and with the proofs by Gelfond and Schneider of that of the general power α^{β}. Some of these results may now be found in several textbooks on the theory of numbers. But until recently there were only two modern books dealing exclusively with the theory of transcendental numbers, viz. that by Siegel (Transcendental numbers, Princeton 1949) and

that by Gelfond (Transcendental and algebraic numbers (in Russian), Moscow 1952).

The present book forms then a welcome addition to this number, particularly so for the rich contents alone of its second chapter. There the author reproduces his general theorem on the algebraic dependence of pairs of meromorphic functions of bounded order the derivatives of which assume algebraic values at finitely many given points. This theorem allows him to deduce in a few lines the transcendency of e and π, that of α^β, and in particular his own important theorems on the transcendency of elliptic functions and integrals and of the modular function.

The other four chapters of the book contain equally interesting and in part very recent material. The high point of the first chapter is a slight generalisation of Roth's recent theorem on the approximation of algebraic numbers by rationals. This result is applied to the construction of simple non-Liouville transcendental numbers.

The third chapter contains the classification of transcendental numbers into the three classes S, T, and U due to Mahler and Koksma. This connects with the measures of transcendency given in the fourth chapter, in particular for e, π, and α^β.

The final chapter deals with Siegel's method of proving the transcendency of solutions of linear differential equations, and it contains his results on Bessel functions.

The book ends with a very useful list of classical and recent literature on the subject.

As this short description already suggests, the book makes an excellent introduction to the theory of transcendental numbers, the more so since the text is clear and readable. K. Mahler

An introduction to Diophantine Approximation. By J. W. S. Cassels. Cambridge tracts in mathematics and mathematical physics, No. 45. Pp. 166. 22s. 6d. 1957. (Cambridge)

Since J. F. Koksma's classical report (Diophantische Approximationen, Ergebnisse der Mathematik IV, 4, Berlin 1936) this is the first book dealing exclusively with Diophantine approximations in the real field; and it forms a valuable addition to the rather small library of books on this subject.

The theory of Diophantine approximation studies the approximate solution in integers or rational numbers of systems of indeterminate equations (in particular, linear ones). Thus the simplest problem demands the finding of rational numbers p/q with small $q>0$ that are close to a given real irrational number θ. The problem may be solved by means of continued fractions, by means of Dirichlet's Schubfachprinzip, or by means of Minkowski's theorem on linear forms; all three methods lead to the result that there are infinitely many p/q satisfying

$$\left|\theta-\frac{p}{q}\right|<\frac{1}{q^2}.$$

Next one can study in what manner these solutions depend on the special character of θ and whether better results can be obtained when θ is suitably restricted.

The present book deals with this problem and much more general ones and discusses the general methods that may serve for the study of the latter.

In Chapter 1, the continued fraction for θ is obtained from the "best" approximations p/q of θ and used to prove Hurwitz's theorem on

$$\left|\theta-\frac{p}{q}\right|<\frac{1}{\sqrt{5}q^2}.$$

Markoff's stronger result is proved in Chapter 2 in a new way that does not use continued fractions, but instead is based on an "isolation" theorem. Chapter 3 deals with inhomogeneous problems, *e.g.* with Kronecker's theorem on the solution of systems like

$$|q\theta_i - p_i - \alpha_i| < \epsilon \qquad (i = 1, 2, \ldots, n).$$

Weyl's related theory of uniform distribution (mod 1) is given in Chapter 4. Next we find "transference" theorems in Chapter 5, *i.e.* theorems connecting pairs of Diophantine problems. Chapter 6 brings Roth's recent improvement of the Thue-Siegel theorem: if θ is an algebraic and ϵ a positive number, then there are only finitely many p/q such that

$$0 < \left|\theta - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}.$$

Chapter 7 contains the metrical theory, with results that hold for "almost all" values of the parameters (in the sense of the Lebesgue measure); *e.g.* the inequality

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{q^2 \log q}$$

has infinitely many solutions p/q for almost all θ, but

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{q^2 (\log q)^2}$$

has not.

A non-linear problem is treated in the final chapter on the Pisot-Vijayaraghavan numbers; these are algebraic numbers θ for which the distance of θ^n, where $n = 1, 2, 3, \ldots$, from the nearest integer tends to zero.

Three appendices collect results from algebra and geometry of numbers that are made use of in the text.

The book is written clearly and should offer little difficulty to an older student. However, he may sometimes wonder why a certain method in the text works, *e.g.* that of the second chapter. For reasons of space, the author has little to say on this question!

K. MAHLER

THE MATHEMATICAL ASSOCIATION

The fundamental aim of the Mathematical Association is to promote good methods of Mathematical teaching. Intending members of the Association are requested to communicate with one of the Secretaries. The subscription to the Association is 21s. per annum and is due on January 1st. Each member receives a copy of the *Mathematical Gazette* and a copy of each new Report as it is issued.

Change of address should be notified to the Membership Secretary, Mr. M. A. Porter. If copies of the *Gazette* fail to reach a member for lack of such notification, duplicate copies can be supplied only at the published price. If change of address is the result of a change of appointment, the Membership Secretary will be glad to be informed.

Subscriptions should be paid to the Hon. Treasurer of the Mathematical Association.

The address of the Association and of the Hon. Treasurer and Secretaries is **Gordon House, 29 Gordon Square, London, W.C.1.**

NEW SOUTH WALES BRANCH

Report for the Session 1957–58

The annual general meeting of the N.S.W. Branch was held on 31st October 1958, when the retiring President, Miss I. Barnes, presented the Presidential address on the topic " The Changing Place of Mathematics in the Secondary School Curriculum ".

Election of office-bearers for 1958 :

President, Dr. I. S. Turner ; *Secretaries*, Miss D. Brownell and Mr. A. R. Bunker ; *Treasurer*, Mr. J. H. Veness.

Other meetings held during 1958 were :

1. 2nd May—Discussion of Mathematics Examination papers.
2. 4th July—Address by Professor Robinson (Toronto) on " Mathematics in Canadian Schools and Universities ".
3. 1st August—Address by Dr. Wall (Sydney University) on " The Galois Theory of Equations ".
4. 19th September—Address by Mr. Graham on the topic " The Value of Exactness in Arithmetic ".

As at 30th September 1958, 800 names were recorded as members.

A. R. Bunker, *Hon. Sec.*

EXETER BRANCH

Report for the Session 1957–1958

Five meetings were held during the year in the Library of the Institute of Education. The subjects were as follows :

" Obstacles to the learning of Arithmetic in Primary Schools " by Mr. L. W. Downes of St. Luke's College, Exeter.

" Musical Logarithms " by Mr. E. H. Lockwood of Felsted School—illustrations provided by strains from the double bass.

" The Logical Structure of School Mathematics " by Mr. W. J. Langford President of the Mathematical Association 1957–58.

" Mathematics in the Aircraft Industry " by Mr. D. C. Wicks of Bristol Aircraft Limited.

" Robert Recorde, 16th Century Mathematician " by Mr. A. Prag of Westminster School.

There are 60 members of the Branch and 24 of these are members of the Association. Meetings have been well attended in spite of some members having to travel considerable distances to get to Exeter.

Officers for the Session have been as follows :

President, Mr. A. P. Rollett ; *Vice-President*, Professor T. Arnold-Brown ; *Secretary*, Miss N. A. Comerford ; *Treasurer*, Miss L. G. Button.

N. A. Comerford, *Hon. Sec.*

NOTTINGHAM AND DISTRICT BRANCH

Report for the Session 1957–1958

The Annual Meeting was held on 30th November 1957, at the Institute of Education. The Secretary's and Treasurer's reports were received and officers elected for the year after which short talks were given by several members. Mr. Swaby spoke on " An Approach to Mathematics in the Lower School " and outlined various ways of introducing topics so that the results are discovered by the pupils for themselves. Dr. Power gave a short introduction to the use of Laplace Transforms with the suggestion that there might

be a place in Advanced Sixth Form Mathematics for this kind of work, Dr. Jackson posed a problem for members to puzzle over, and Mr. Chettle spoke on " End of Term Activities " giving suggestions for work on loci and envelopes which could be done in the Middle School and which would prepare the way for more formal treatment later.

The Spring Term meeting on 8th March again took the form of a Day Conference held at Nottingham University. The Speaker at the Morning Session was Dr. Jackson of Liverpool University on " Friction ". Dr. Jackson showed that the direction of the frictional force was not always in the direction opposing relative motion if no friction existed, and that in some cases the direction of friction changed discontinuously when motion began so that arguments based on initial motion were fallacious. He demonstrated that frictional problems could be solved without any preconceived ideas about direction. The first afternoon session was addressed by Mr. Bushby of the Meteorological Research Department on " The Use of Computers in Forecasting ". Mr. Bushby explained the equations used for forecasting pressure distribution, and indicated how they could be solved using a relaxation method. The computer did in 32 hours work which would take 4–5 years by hand, and the results obtained agreed well with the actual pressure distribution.

Dr. Haslam-Jones of Queens College, Oxford then spoke on " The History of Differentiation since 1800 ", selecting this period as one in which formal analysis without much regard to rigour was replaced by the rigorous approach of the modern mathematician. He traced the changes in the approach to the problem of the graph of a continuous function to illustrate the change.

The Summer Meeting was held at Nottingham High School on 21st June when Dr. Adkin of Nottingham University spoke on " Rubber in Mathematics ". He discussed the molecular Structure of rubber and investigated the most probable separation of the ends of a molecular chain. He also discussed the research into the problem of finding a more accurate result than Hooke's Law for measuring tension in terms of extensions and showed how various formulae were obtained and tested.

The officers for the year were :

President, Dr. G. Power ; *Vice-President*, Mr. K. R. Imeson ; *Secretary*, Mr. F. E. Chettle ; *Treasurer*, Mr. C. R. Swaby.

F. E. Chettle, *Hon. Sec.*

SOUTHAMPTON AND DISTRICT BRANCH

Report for the Session 1957–1958

During the 1957–1958 session there were six meetings of the Branch. These were held in the different Grammar Schools of Southampton as well as at Southampton University and Winchester College. 33 Members and 30 Associates (half the latter being students) form the official membership of the Branch, but the average attendance of 44 also includes visitors who see the meetings advertised in the Local Education Circular.

Friday, 25th October, 1957. At the Annual General Meeting of the Branch, the following were elected to the Committee for 1957–58 : *President :* Dr. L. J. Stroud, *Vice-President :* Mrs. W. S. P. Edmunds, *Secretary and Treasurer :* Mr. J. C. F. Fair, *University Representative :* Miss N. Walls, also Miss R. A. Clarke, Mr. T. A. Jones and Mrs. M. Laing. The rules of the Branch having been revised were accepted by the meeting. Mr. W. Hope-Jones, a past-president of the Association, gave a stimulating talk entitled " Some Fun with Probability ", in which he considered several amusing problems of chance which often gave unexpected results.

Thursday, 25th November, 1957. Dr. H. J. Davies of Southampton University gave a lecture about " The Jet Flap ". He traced the development of lifting devices and differing types of aircraft flaps, culminating in their possible replacement by a jet stream, as he demonstrated in wind-tunnel experiments.

Friday, 31st January, 1958. Dr. R. S. Scorer, from Imperial College, London, gave a talk on " Mountain Waves in the Atmosphere " illustrated by some excellent colour slides. Stationary wave clouds were shown to be of a different nature from the ordinary convection formations.

Thursday, 4th March, 1958. Professor E. T. Davies of the University of Southampton opened a discussion on " The Transition from School to University Mathematics ". Just as there were different views on the purpose of Mathematics at the various stages, so there were various ideas about the amount of teaching desirable and the depth of rigour required at the sixth form level.

Monday, 12th May, 1958. This meeting was held jointly with the Department of Mathematics in the University and the University Mathematical Society, and was attended by nearly 100 people (this number is not included in the Branch average above). Professor C. B. Allendoerfer, a Fulbright Scholar at Cambridge University, from the University of Washington, delivered a lecture on " The Training of Mathematicians in the United States ". The main differences from the United Kingdom he saw to be the grading of the pupils by a teachers assessment as opposed to our examinations, and the completely separate teaching of the separate branches of mathematics as compared with our parallel or unified courses.

Friday, 20th June, 1958. A brief report of the Annual Meeting of the Association held at Easter in Manchester was given by Miss P. M. Pickford. Then Mr. K. F. Solloway, from Leicester, gave a demonstration lecture entitled " Manifestation—or—See for yourself ". His theme was that Mathematics should be learnt by understanding rather than by rote. His enthusiastic use of helpful blackboard illustrations was infectious.

J. C. F. Fair, *Hon. Sec*

VICTORIA BRANCH

Report for the Session 1957–1958

This year has again been an active year under the presidency of Miss Margaret Lester.

The Committee has met ten times and there have been eight general. meetings and two special meetings.

Attendance at the general meetings was very pleasing with three general meetings having an attendance of over 200.

The general meetings were as follows :

1.	Leaving Mathematics I and II	Professor E. R. Love
2.	Mathematics in the Geophysical Year	Professor M. Cherry
3.	Statistics in the Services	Mr. R. Metzenthen
4.	Towards Developing the Mathematical Imagination of the School Pupil	Miss M. Lester
5.	The Cuisenaire—Gattegno Method of Teaching Arithmetic	Mr. K. Evans
6.	Schools Afternoon :	
	(*a*) Irrationals in Music	Mr. R. Harrison
	(*b*) Switching Algebra	Mr. J. Ryan
7.	A Statistical and Historical Analysis of the Transferable Vote	Mr. C. H. Allen

8. Annual Meeting

The Special Meetings were :

1. The Place of Projective Geometry in a Liberal Education — Professor G. de B. Robinson of Toronto University
2. The Cuisenaire-Gattegno Method of Teaching Arithmetic — Dr. G. Gattegno

The Report on the Recruitment of Mathematics Teachers has been followed up with several discussions with the various authorities.

The introduction of a B.Sc. course with Mathematics as major subjects is one result of the report.

The first of a series of articles designed to assist those teachers of mathematics who are not trained for the purpose has appeared in the University's Circular to Schools. It is hoped that this article will be followed by many more and that members who will be approached by the Committee will co-operate by writing suitable articles to assist this group of teachers. Mr. T. C. Keating has agreed to edit these articles and was co-opted to the Committee on July 24th, 1958, for this purpose.

During the year, financial support was given to the Decimal Currency Council.

This Association was invited to nominate a representative to the Science Liaison Committee. Mr. G. R. Clark was nominated and has ably defended the mathematics viewpoint at the meetings, for in particular, this Association is deeply concerned regarding the possible results if the suggestion were adopted that the number of Mathematics subjects at Matriculation level be reduced.

H. J. Russell, *Hon. Sec.*

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